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Option Pricing with GOU Process under a Stochastic Earning Yield

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Declaration

This thesis contains no material which has been accepted for the award of any other degree or diploma in any university.

To the best of my knowledge and belief this thesis contains no material previously published by any other person except where due acknowledgement has been made.

Signature: __________________________

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Abstract

Since the classical Black-Scholes model was introduced in early 1970s, it has been used as a primary model for option pricing in financial markets. The aim of this research is to further study and extend the Black-Scholes models to take into account more factors and features of the financial market in the model. The research consists of three major parts, including various alternative derivation of the Black-Scholes models under different application perspectives, developing a European option pricing model taking into account earning yield of stochastic nature, and investigating the performance and application of the new model through numerical simulation and comparison with the existing models and real option price data.

Many approaches have been proposed to derive the Black-Scholes model. Through examining the basic assumptions made and the existing approaches used, three derivations of the Black-Scholes type models are developed under different application perspective. The first derivation is based on application with different asset types including long-term assets and current assets. The second derivation concerns a portfolio with n stochastic assets and one deterministic asset, while the third derivation deals with assets with a constant dividend filed.

The second part of the work is to develop a European option pricing model taking into account dividend yield and earning yield of stochastic nature. The dividend yield is assumed to follow the generalized Ornstein-Uhlenbeck (GOU) process with a stochastic mean-reverting earning yield. The existence of three stochastic components of the dynamic GOU process allows the dividend to randomly deviate from the earning yield flow. Stock prices, dividend yield and earning yield are modeled under a stochastic mean-reverting market price of risk (MPR) environment. Explicit formulas are then derived for the European call and put

option prices.

The third part of the research is to validate the new option price model. Firstly, explicit formula for various Greek parameters are derived to study the sensitivity of the option price model, such as the rate of change of the option price to the change in the asset or stock price or to the change in the implied volatility. Then numerical experiments are carried out to analyze the performance of the model by checking the sensitivity of option price to each of the model parameters. Finally , through numerical simulation, the overall performance of our new model is compared with that of various major existing option pricing models and real option price data, and our results show that our new model out-performs others.

List of Publications Related to this Thesis

1. N. Phewchean, Y.H. Wu, Option pricing taking into account stochastic earning yield. Part I: Theoretical formulation, under review
2. N. Phewchean, Y.H. Wu, Option pricing taking into account stochastic earning yield. Part II: Numerical investigation, under review

Abbreviations

OU	Ornstein-Uhlenbeck
GOU	Generalization of Ornstein-Uhlenbeck
MPR	Market Price of Risk
CBOE	Chicago Board Options Exchange
BSC	Black-Scholes Call Option
CDC	Constant Dividend Black-Scholes-Merton Call Option
SDC	Stochastic Dividend Yield Call Option
SEC	Stochastic Earning Yield Call Option
BSP	Black-Scholes Put Option
CDP	Constant Dividend Black-Scholes-Merton Put Option
SDP	Stochastic Dividend Yield Put Option
SEP	Stochastic Earning Yield Put Option
DJX	1/100 Dow Jones Industrial Average
SNP	S&P 500
RUT	Russell 2000
NDX	NASDAQ-100

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Chapter 1

Introduction

1.1 Motivation and Objectives

In the world economics, assets are resources which can be categorized into two main categories, tangible and intangible assets. In brief, tangible assets could be described as assets which can be physically measured or touched. On the other hand, financial assets are the enormous example of intangible assets which are playing a more important role in the financial market than tangible assets because of more liquidity property.

A financial asset is a contractual claim on an economic unit which can derive the value. Stocks are playing the most significant role in the investment market. They are issued by a company and can be sold by one individual to the others. Investors are assured by a company to purchase multiple shares, divided from the stock of a company, so that a company can gain equity capital [1].

In recent years, derivative products such as futures and options have increasingly become significant tools for investors throughout modern world market. Stock option trading began with the 1973 establishment of the organized exchange, Chicago Board Options Exchange (CBOE) besides stock futures. Options are now traded in a huge volumes on many exchanges throughout the world especially in high impact investment countries [2]. In this thesis, mainly stock option is focused since stock option is one of the most innovative and flexible derivative.

In 1973, Fischer Black and Myron Scholes contributed significantly to the development of option exchange. They published the famous article, "The Pricing of Options and Corporate Liabilities", in the Journal of Political Economy. The classic Black-Scholes formula is an equation for calculating option prices based on the financial parameters of stock price $S(t)$, strike price $K(t)$, volatility σ, interest rates r , time t and maturity T [3]. The Black-Scholes model was applied, on an assumption of non-dividend paying of the stock, to valuate European option pricing. However in reality, most companies pay a return to investors in the form of dividend. Consequently, at the same year, Robert C. Merton proposed the extended Black-Scholes model to incorporate a known annual dividend yield in order to conquer the assumption of non-dividend paying and, as a result, the well known Black-Scholes-Merton model was developed to include the dividend yield parameter $\delta(t)$ [4–9].

Hence, motivation of this research was first influenced by the concept of new parameter addition of the extended option pricing model of Merton [5]. Under the option pricing theory, many factors of option pricing have been included in recent option pricing models. The new specific process or function or even the combination of processes or functions for the parameters under the underlying asset return were also proposed during the last few decades after the classic Black-Scholes model formation in 1973. Options pricing were evaluated by many other approaches such as general equilibrium or the no arbitrage arguments approach, including econometric method to construct econometric models [10–37].

An idea has been enlightened by everyday-newspapers within the stock and investment section. Dividend is the parameter which has been discussed in many articles under the topic of Options Pricing [9, 38–43]. Next to the dividend column, there is always a P/E ratio column which gathers one of the most important information for the investors since the P/E ratio of a stock is one of the most important key tools for the investor to make a decision to buy or sell a stock. Consequently, the P/E ratio should be a parameter in the option pricing model in order to make the model of option pricing more precise. Some works under the P/E ratio consideration have been published, for example, Gurdip Bakshi and Zhiwu Chen proposed the idea in the article titled "Stock valuation in dynamic economies" to model a stock valuation by considering earnings of the firm instead of dividends [44], while Ming Don and David Hirshleife proposed another method to generalize the idea of Gurdip Bakshi and Zhiwu Chen [45]. By obtaining the prospective of the work of Gurdip Bakshi and Zhiwu Chen including the other articles [46, 47], the P/E ratio is a significant factor in order to develop a model to price stocks or options.

In order to construct the model under the P/E ratio parameter, we need suitable assumptions based on the reality of finance. Stochastic process is one of the recent mathematical behavior which suitably describes the behavior of stock or option price. In the evolution of stochastic process, the Ornstein-Uhlenbeck process has regulated the mathematical model to valuate the option price [38, 39, 48]. Together with an idea to generalize the process of Ornstein-Uhlenbeck, we sum up all the ideas to come up with this thesis under the name of "Option Pricing with GOU Process under a Stochastic Earning Yield".

1.2 Contributions

We propose this thesis by generalizing the Ornstein-Uhlenbeck process and give the title of the new process as GOU process. By the method of Abraham Lioui to improve the non-deterministic model into deterministic formula [38], we derive the models for option pricing under the GOU process based on earning yield. We also propose the alternative approaches for derivation of the Black-Scholes-Merton Models, and by this concept we bring an idea to prove for Greek parameters as an proposition. For option pricing we test our model and the results show that the model has the same trend of pricing as the other classic models proposed by Black, Scholes and Merton [3], including the other extended option pricing models [4, 5, 7–9, 12, 38].

The real data of financial world has also been applied to the model in order to do empirical testing. We compare our model to the previous proposed models including the classic Black-Scholes model with the volatility parameter determined by the least-squared error method. By choosing proper parameters, the empirical results show that our model has significantly less error than the other previous models.

The fact that the P/E ratio or the reciprocal of the earning yield affects the price of the option is taken into account in this study. Moreover, the earning yield parameter is shown to be another key parameter which has an immense impact on the option pricing model.

1.3 Outline of the Thesis

In the next chapter of this thesis, Chapter 2, we will briefly discuss the details of option pricing models. The background knowledge of this field will be explained abruptly including previous research works for option pricing. The purpose of this chapter is to show why we need to continue this research as we propose in this thesis. In Chapter 3, we will study the approaches to prove the classic Black-Scholes-Merton model. We propose alternative derivation of the model. This chapter will bring the idea to derive the proposition in Chapter 5, Greek Parameters . In Chapter 4, we will clearly explain how to derive the option pricing model by using the concept of new stochastic process, the GOU process. Under the stochastic earning yield assumption, the option pricing model is derived. The proposition of the new model has been tested. The propose of this chapter is to propose the existence of new stochastic process and the new model of option pricing. In Chapter 5, Greek Parameters have been studied and proposed as propositions by analytical derivation. After we propose the new model, in Chapter 6 we investigate the performances and characteristics of our extended option pricing models as well as observe the sensitivity of parameters. In Chapter 7, we demonstrate that the new option pricing model is applicable to the real world. The real world financial data has been used to test the model. The empirical results have been discussed. The propose of the chapter is to compare our model to the other previous models and show that our model is more applicable and better than the existing pricing models. Lastly, the final chapter, Chapter 8, is the chapter to conclude all of our results in this thesis and discuss the way to further our work in the future.

Chapter 2

Literature Review of Option Pricing Models

2.1 General

In this chapter, we briefly review the background knowledge highly relevant to the research on option pricing models. We particularly review previous research works and literatures of option pricing models and analysis of their basic characteristics and behaviors. The chapter will also highlight and explain the reasons of doing this thesis: why we need to study another parameter in this research.

In 1973, Black and Scholes introduced some new financial concepts for derivative pricing models and the established Black-Scholes Model has since then become the standard basic model for pricing options in the financial market. The Black-Scholes model has been used widely in practice for an enormous variety of financial derivatives such as options and bonds. Researchers in financial mathematics field have also given more attention in the option pricing theory [10–37]. In this research, we will specifically focus on the option pricing model.

2.2 Option

In financial markets, derivative products are traded as the modern security for the wise investors. A financial derivative is named because of the reason that it is "derived" from the underlying asset price such as a stock, a stock index, a commodity, a foreign currency and etc. One of the most important derivatives which plays a very significant role for the investors is option, as it is a tool to decrease the risk of a portfolio in such an investment.

Options can offer investors the flexibility in any investment circumstance which the investors might encounter. Options provide the trader options. With options, investors have the freedom of trading for their position and situation. By holding the option contracts, the investors will have various benefits; for example, stock holdings will be protected from a decline in stock market, the value of portfolio may increase with the option holdings, the investors have the right to buy stock at a lower price, the traders can position themselves for a risky fluctuated situation in such an uncertain movement of the stock market, the incomes from trading may increase without the cost of purchasing the stocks, and etc. [49]

Individual investors who wish to buy or sell options can place an order through their brokerage firms. The options will be traded on which point based on both the policy of the brokerage firm and the type of the option contract of exchange(s) to be traded. Nowadays, options are traded on: The Chicago Board Options Exchange, Inc. (CBOE), the American Stock Exchange, Inc. (AMEX), The Pacific Stock Exchange, Inc. (PSE), the New York Stock Exchange, Inc. (NYSE), the Philadelphia Stock Exchange, Inc. (PHLX), and including the foreign markets such as the Australian Security Exchange (ASX). [49, 50]

2.2.1 Option Definition

Particularly, option is a contract to buy or sell an underlying asset or any asset combination. Since the famous Black-Scholes-Merton model for pricing plain vanilla option was introduced in more than three decades ago, the theory of option pricing has been developed by many researchers and then led to the innovation of the new types of options and, subsequently, the modern theories of option pricing.

There are two basic options types, call option and put option. By definition, a call option is a contract which provides holder the right, but not the obligation, to purchase the underlying asset such as a stock at a specified price on or before a given date. Conversely, a put option is a contract which gives the holder the right, again not the obligation, to sell the underlying asset such as a stock at a specified price on or before a given date.

If the option must be exercised on a given date, the option will be the so called "European option"; whereas if the option can be exercised before a given date which provides more flexibility to the investors, this type of option is defined as "American option" [1, 2, 51].

In this thesis we will specifically pay an attention on the European option and mostly with the call option as it plays more important role in mathematical consideration.

2.2.2 Option Factors

There are several key elements which affect the option price. It has been well known that there are six factors which mathematically have an influence on the stock option prices: the underlying stock price (S), the exercise price or strike price (K), the time to maturity or the time to expiration (T), the risk-free interest rate (r), the volatility of the stock price (σ) and the dividend yield (δ) [2].

Stock Price (S) is the parameter to determine the payoff for a call option with a strike price K which can be represented mathematically as max $\{S(t) - K, 0\}$. By this mathematical expression, the value of call option will be increasing when the stock price is increasing while the value of call option will be decreasing when the stock price is decreasing. If the stock price is higher than the strike price K at maturity T, the pay off is $S(t) - K$ because the buyer has the right to buy a stock at the expiration time at the strike price which causes the buyer to make a profit $S(t) - K$ when he/she sells the stock immediately. Conversely, if the stock price is less than the strike price K at maturity T, the pay off will become zero since the buyer can not make any profit from this situation. Likewise, the payoff for a put option with a strike price K can be represented mathematically as max $\{K - S(t), 0\}$. By the similar explanation, the put option is more valuable as the stock price decreases and it will be less valuable when the stock price increases.

When the strike price K is higher than the stock price at maturity T, the pay off is $K - S(t)$, otherwise the value of payoff will be zero.

Strike Price (K) is the specified price which is written in the contract and can be traded when the option is exercised. From the definition of payoff of the call option, $\max\{S(t) - K, 0\}$, when the strike price is increasing, the call option value is decreasing; conversely, the put option price is increasing and vice versa.

Maturity (T) is the time to expiration of the option. The longer time to expiration, the more opportunities to exercise the option. Therefore, generally, the price of the option increases as the maturity increases. However, this is just the theoretical fact. In reality the value of the option can be fluctuated as a result of the movement of the market including the demand of investment.

The volatility (σ) is a measure for uncertainty of the parameter. Generally in finance, volatility mostly refers to volatility of the stock price movements (σ_S). Large number of volatility can affect the wide variation of the stock price. This implies that a larger value of volatility may cause larger variation of option price.

The risk-free interest rate (r) is the theoretical rate of return of an investment when there is no risk of financial market. In the other words, the risk-free interest rate can refer to risk-free rate of return as the rate that an investor would expect from the investment in an completely risk-free market over a given time period. Since the risk-free interest rate can be acquired when there is no risk, this implies that the investment with any additional risk should be returned by an interest rate which is higher than the risk-free rate. Typically, when the interest rate increases, this will cause the price of call option to increase while the put option to decrease. In contrary, if the interest rate decreases, the call option value will decrease and put option value will increase.

Dividend yield (δ) is the firm's total annual dividend payments which is divided by its market capitalization. In other words, it is represented as the percentage and known as the dividend per share, divided by the price per share. Generally, the dividend payout affects the decrease of call value and the increase of put value.

In summary, the above are the six main factors for the consideration of option pricing formula by the researcher [52]. In this thesis we will pay our interest to the parameter addition to the option pricing model and explain the possibility of a better option pricing model.

2.2.3 Put-Call Parity

In financial mathematics, put-call parity represents a relationship between the price of a European call option and European put option in a financial market assuming no transaction costs by setting the other parameters in an option pricing model as the same value. It is well known for the Black-Scholes model for the put-call parity property. In this thesis, we will use the property of the put-call parity in order to find the value of European put option once the European call option is derived [52].

2.2.4 Stock Indices Options

As its name says, the stock index option is a financial derivative or a type of options which provide the investor the right, but not obligation, to purchase or sell a basket of stocks, such as the well known S&P 500 [53].

Stock indices options are traded in both the over-the-counter and exchange markets. Because of the large amount of the stocks to be considered for one index, a stock index can perform as a track of the movement of the large amount of the stocks in which the index refers to as a whole. Some indices are based on the movement of the particular sector of stocks such as technology, transportation, utilities and etc.

For example, one of the most famous stock index is S&P 500 or the Standard and Poor 500 which is the index to measure the whole stock prices of 500 large-cap stocks based in the United States. The options for S&P 500 are European and trading on the Chicago Board Options Exchange (CBOE).

Index options are settled in cash. This implies that the holder of a call option receive max $\{S(t)-K,0\}$ in cash and the option writer pays this amount in cash as well. Same concept applies to the put option, the holder of a put option receive max $\{K-S(t),0\}$ in cash. The cash payment is based upon the value of the index at the end of the day. Each contract, the value of the index is multiplied by the multiplier of each contract, for example the multiplier of S&P 500 is \$100. [2, 49]

2.2.5 Moneyness

In finance, moneyness for a European option is a measure and is defined as the ratio between the stock price S, and the strike price K. Mathematically, it can be written as S/K. If S is greater than K, the option is said to be in-the-money (ITM); if the S is less than K, then the option is said to be out-of-the-money (OTM); and if S is equal to K, the option is at-the-money (ATM).

2.2.6 Greek Parameters

In mathematical finance, the Greek parameters, also known as risk sensitivities or hedge parameters, are the numerical quantities showing the sensitivities of the derivative prices such as the option prices to a change in the underlying parameters on which the value of the derivative is dependent. The changes or the sensitivities to parameters are denoted by Greek letters.

The Greek parameters are the significant tools to manage risk for the investors. Each parameter of Greek can observe the sensitivity of the portfolio value to the change of an underlying parameter. In the Black-Scholes model, the Greeks parameter are obtained easily by simple mathematical approach and these parameters are useful for the derivatives traders.

Mathematically, these can be derived by the derivatives. First order derivatives are the most common Greek parameters for example, Delta , Vega, Theta and Rho. Gamma is an example of a second-order derivative of the value function [54]. There are many Greek parameters in option pricing theory as shown in table 2.1.

Table 2.1: List of Greek Parameters

Greek Parameter	Definition
Delta	The rate of change of the value of an option with respect to change in the stock price
Theta	The rate of change of the value of the option with respect to time
Rho	The rate of change of the value of the option with respect to rate of interest
Vega	The rate of change of the value of the option with respect to rate of volatility
Gamma	The rate of change of the value of the delta with respect to change in the stock price
Vanna	The rate of change of the value of the option once with respect to change in the stock price and once with respect to volatility
Vomma	The rate of change of the value of the vega with respect to rate of volatility
Charm	The rate of change of the value of the delta with respect to time
DvegaDtime	The rate of change of the value of the vega with respect to time
Vera	The rate of change of the value of the rho with respect to volatility

In this thesis we will study the characteristics of some of the important Greek parameters including Delta, Gamma and Vega for testing the sensitivity of stock price to the option value, the sensitivity of stock price to the delta and the sensitivity of volatility to the option price respectively as well as to verify the behavior of our proposed option pricing model for stochastic property.

2.3 Wiener Process

In mathematics, the Wiener Process is a continuous time stochastic process. It is also well known as the standard Brownian motion. This stochastic process has been applied in the application of mathematics for decades. More importantly, it has played an important role in stochastic calculus which is the new key of the modern research world. Some applications of Wiener process are the Gaussian white noise process with the application of the noise model in electronics engineering and also for mathematical finance with an application to financial model as it represents the movement of the market.

The Wiener process, $W(t)$ has three properties [55]:
(1) $W(0) = 0$
(2) $W(t)$ is almost surely continuous.
(3) $W(t)$ has independent increments with $W(t) - W(s)$ follows the normal distribution $N(0, t - s)$

where $0 \leq s < t$ and $N(\mu, \sigma^2)$ is normal distribution with an expected value μ and variance σ^2.

This Wiener process also has the vital properties as follows.
(1) $dt^2 = 0$
(2) $dW(t)dt = 0$
(3) $dW(t)^2 = dt$

One of the most significant applications is a stochastic process defined as

$$X = \mu t + \sigma W(t) \tag{2.1}$$

which is a Wiener process with drift μ and variance σ^2. The simple explaination of this process is the process which is composed of the certain movement with the average drift of μ; however, with the property of uncertainty in this stochastic process, the term $W(t)$ represents the stochastic term with the variance σ^2. From equation (2.1), a generalized Wiener process can be explained by an extension of an ordinary differential equation (ODE) to a stochastic differential equation (SDE) derived by adding a normally distributed random variable into the ODE as follows [55, 56].

$$dX = adt + bdW(t) \tag{2.2}$$

where a and b are constants and $dW(t)$ is a normal random variable.

This Wiener process has brought out the basis of the financial process and the standard to prove the Black-Scholes model as it will be shown in Chapter 3.

2.4 Ito's Lemma

In recent years, Kiyoshi Ito has discovered a particular type of stochastic processes and it was named as Ito's process where Ito's lemma is applied in. Ito's lemma is particularly used in Ito stochastic calculus for solving the stochastic processes. Nevertheless, the lemma is widely applied into the application of financial mathematics. One of the most well known applications is to derive the Black-Scholes equation for option pricing.

From equation (2.2), Ito has set the generalized version of the process and represented the Ito's process as

$$dX = a(X,t)dt + b(X,t)dW(t) \tag{2.3}$$

where $a(X(t),t)$ and $b(X(t),t)$ are functions of t and the random process $X(t)$. Ito's lemma describes specifically the process with a function of a Wiener process and assumes that there is a function $F(X,t)$ with continuous first derivative F' and continuous second derivative F''.

From Taylor's theorem, we have

$$
\begin{aligned}
dF(X,t) &= F_t dt + F_X dX + \frac{1}{2}F_{tt}(dt)^2 + \frac{1}{2}F_{XX}(dX)^2 + F_{tX}dtdX \\
&= F_t dt + F_X\big(a(X,t)dt + b(X,t)dW(t)\big) + \frac{1}{2}F_{tt}(dt)^2 \\
&+ \frac{1}{2}F_{XX}\big(a(X,t)dt + b(X,t)dW(t)\big)^2 \\
&+ F_{tX}dt\big(a(X,t)dt + b(X,t)dW(t)\big)
\end{aligned} \tag{2.4}
$$

Since $dW(t)$ has the properties

(1) $dt^2 = 0$,

(2) $dW(t)dt = 0$,

(3) $dW(t)^2 = dt$,

from equation (2.4), we have

$$
\begin{aligned}
dF(X,t) &= F_t dt + a(X,t)F_X dt + b(X,t)F_X dW(t) + \frac{1}{2}b(X,t)^2 F_{XX}dt \\
&= \big(F_t + a(X,t)F_X + \frac{1}{2}b(X,t)^2 F_{XX}\big)dt + b(X,t)F_X dW(t).
\end{aligned} \tag{2.5}
$$

The equation (2.5) represents the Ito's lemma which is very useful for the application in financial derivative theory [57].

2.5 Ornstein-Uhlenbeck Process

One of the most applicable stochastic processes, Ornstein-Uhlenbeck process, is a stochastic process which can be easily described as the movement of a massive Brownian particle under the friction influence. It is also well known as mean-reverting process.

The Ornstein-Uhlenbeck process, $X(t)$, is defined as a stochastic differential equation:

$$
dX(t) = \theta(\mu - X(t))dt + \sigma dW(t) \tag{2.6}
$$

where the parameter θ, μ and $\sigma > 0$ and $W(t)$ is the Wiener process. The parameter θ can be performed as the friction coefficient.

One of the well known applications for Ornstein-Uhlenbeck process is a Hookean spring. Also, in financial mathematics, it is represented as a stochastic model for interest rates, currency exchange rates, etc. The parameter μ is the mean value, σ is the volatility and θ is the rate for mean reverting. Another example is pairs trade strategy [58].

There are many researchers who have been using the Ornstein-Uhlenbeck process for financial mathematics filed of research, since the process can describe the characteristic or the behavior of the subject. For example, the Ornstein-Uhlenbeck process can describe the stochastic volatility by R. Schobel and J. Zhu. [59] and by E.Nicolato and E.Venardos [60] or to apply for stochastic dividend yield by [9, 38, 39]. These are just a few examples since there are a huge number of the applications from this useful process [61].

In this thesis we not only apply the Ornstein-Uhlenbeck process but also extend the Ornstein-Uhlenbeck process for the usage of our proposed financial model.

2.6 Option Pricing Models

An option pricing model is a mathematical model which requires some parameters to be involved in order to provide the fair price of the option in the financial market. In 1973, Black and Scholes proposed the nobel article which was an introduction of option pricing model. Then this topic has been researched and extended by many researchers [3]. From the start of Black-Scholes environment, the extensions of the option pricing models have been proposed to describe the dynamics of the parameters in market such as underlying stock, volatility, interest rate and dividend yield. There are three main approaches for modeling the option price: 1) continuous-time finance approach and 2) econometric approach and 3) numerical method [62].

The continuous-time finance approach has been proposed by the assumption of a model or a process or a function to describe the parameters under the option pricing model. For example, the model of Black-Scholes with the extension of a function to describe the dividend yield [4, 5, 7–9]. Jump diffusion is one of the improvement of the models that provides the interesting results of the option prices [63–65]. Some parameters or the movement of the markets also can be described by a specific process [66, 67]. The recent option pricing models are assumed to be the combination of processes of the combination of parameters [68].

The econometric approach applies the econometric models to explain the movement of the market or the complexity of the underlying asset with the econometric properties. The most well known models to price the option in econometric approach are the AutoRegressive Conditional Heteroskedasticity (ARCH) model and Generalized AutoRegressive Conditional Heteroskedasticity (GARCH) model. With the development of the econometric method some researchers proposed the Regime Switching model [10–37].

The last category of option pricing approach is the numerical method. This method generally is simpler than the previous approaches. For instance, the widely known binomial option pricing model provides a generalizable numerical method for option pricing [37]. Continuously, the researchers have also developed the method for pricing the option [69, 70].

In our research, the continuous-time finance approach is applied under the Black-Scholes-Merton framework. We propose the new parameter to play a role in the option pricing model with the extended stochastic process of Ornstein-Unlenbeck process as the new perspective of modern development of the new era of financial world.

2.7 Concluding Remarks

This chapter reviews the background of the option pricing models. The review is relevant to the two main parts that we will discuss later including the derivation of the Black-Scholes model and the Ornstein-Uhlenbeck Process for option pricing model. The basic concepts presented here will play a very significant role for the extension of the Black-Scholes option pricing model in our research.

Chapter 3

Derivation of the Black-Scholes type Option Pricing Models

3.1 General

Option is the derivative that we are to discussed in this research work. A derivative is a financial security whose price is dependent upon or derived from the value of the other underlying assets, such as stocks, bonds, currencies, stock indexes and commodities. There are many kinds of derivatives such as options, futures, forwards and swaps. A derivative may be called a security since the derivative plays a very important role to control the risk of investment in the financial market, such as stock market. The first problem for the study of derivative is how to price or evaluate the derivative in the market. The Black-Scholes model has been introduced in 1973 by Fischer Black and Myron Scholes [3] for this purpose. This elegant formula has changed the derivative pricing view of financial practitioners and theoreticians [54]. Under the security named as European call option, the partial differential equation, the Black-Scholes equation, has performed the aspect to price by taking into account the basic factors such as stock price, exercise price, maturity and interest rate. From this fascinating proposition, Scholes received the Nobel Prize in economics in 1997 (Black had passed away in 1995 before the prize was nominated.) [71].

Many researchers have discussed the derivations of the classic Black-Scholes equation [3–6, 71, 72]. By understanding the concept involved, it can bring out the extension of the model such as Robert C. Merton's work which includes the dividend yield in the model. Many perspectives have been brought to the Black-Scholes equation derivation, for example the concepts of replicating the derivative with a stock and bond, replicating the bond with a stock and a derivative and replicating the stock with a bond and a derivative. By including the alternative derivations such as by using the CAPM, using the return form of arbitrage pricing or using risk neutral pricing method, the Black-Scholes formula can be derived by various different methods [71].

The Black-Scholes-Merton equation is the most well-known model for option pricing. One of the most significant part in the study of the Black-Scholes-Merton equation is the derivation, as we need to understand the principle behind this partial differential equation that governs the price of options, including all other financial securities. The idea of how to derive this model can be seen by many perspectives. Many researchers have investigated on various derivations [71]. In this chapter, we are concerned with alternative methods to derive the classic Black-Scholes-Merton equation. Firstly, we propose another perspective of derivation through the concept of asset type, which can be categorized into two types: current asset and long-term asset. Secondly, we propose another concept of derivation by focusing on a number of assets, n assets. Finally, for the third derivation we propose to include a constant dividend yield D_0 into the model of n assets. The proposed alternative derivations in this chapter will be applied to the proof of the proposition of the new model of option pricing in this thesis. Thus the derivation of this classic Black-Scholes-Merton equation is important and is actually applied through the financial mathematics perspective.

3.2 Derivation Background

By the definition of derivative, the very basic concept to derive the Black-Scholes equation is to replicate portfolio that composes of bond and stock, and to establish the equation by setting the returns of replicating portfolio equal to the payoffs of the derivative. Mathematically, there are many approaches to derive the Black-Scholes or the Black-Scholes-Merton equation. In this thesis we will refer to the replicating portfolio approach since we will use it to prove the Geek Parameters proposition in Chapter 5.

The well-known approach to derive the Black-Scholes formula is to construct an equation to represent a replicating portfolio. A portfolio can be replicated by a risk-free bond and a stock and this portfolio can be treated as the payoffs of the derivative. By this method, there are various explicit assumptions to make in order to derive the basic Black-Scholes model for a particular stock as follows.

(1) There is no arbitrage opportunity which implies that the price of the specified asset will not be different among the markets: investor can not make a profit by this.

(2) A known constant risk-free interest rate is assumed for all maturities.

(3) It is permitted to buy and sell, including short selling, at any amount.

(4) Frictionless market is assumed which implies that there are no transaction costs or fees.

(5) A geometric Brownian motion is assumed for the stock movement with constant drift and volatility.

(6) There are no dividend for underlying security.

(7) Trading in the market is assumed to be continuous.

By considering the fact of investment, the increase amount of investment during the small time interval is equal to the return amount of that investment in that time interval. This fact can be interpreted mathematically.

Let P be the investment amount and r be the fixed rate of the investment return, then the increase amount of investment during the small time interval is equal to the return amount of that investment in that time interval, namely

$$dP = Prdt, \tag{3.1}$$

that is

$$\frac{dP}{P} = rdt. \tag{3.2}$$

If we let $X = lnP$, then the equation (3.2) becomes

$$dX = rdt \tag{3.3}$$

which is the deterministic equation. However, in the financial market, we need to take account the stochastic nature into the equation, and thus a stochastic parameter or term must be considered in the model. Equation (3.3) becomes

$$dX = rdt + \sigma\sqrt{\Delta t}dW(t). \tag{3.4}$$

The stochastic term of $\sigma\sqrt{\Delta t}dW(t)$ can be recognized as a stochastic random variable with zero mean and $\sqrt{\Delta t}$ standard deviation. Equation (3.4) is known as a Stochastic Differential Equation (SDE).

If the term $W(t)$ in the equation (3.4) is the Wiener process, this equation is the stochastic differential equation with the Wiener Process. In other words, it can be described as the ordinary differential equation which contains the stochastic parameter. Equation (3.4) becomes

$$dX = adt + bdW(t) \tag{3.5}$$

where a and b are constants and dW(t) is the stochastic term with the Wiener process $W(t)$. To make the equation more general, the constant term can be changed into the function term and the equation is changed into the general form as follows.

$$dX = a(X,t)dt + b(X,t)dW(t) \tag{3.6}$$

where $a(X,t)$ and $b(X,t)$ are functions of X and t. This equation is called Ito's Process.

By Ito's lemma, if we set the random variable $F = F(X,t)$, from the equation (3.6), we can derive

$$dF(X,t) = \big(a(X,t)F_X + F_t + \frac{1}{2}b(X,t)^2F_{XX}\big)dt + b(X,t)F_X dW(t) \tag{3.7}$$

To derive equation (3.7), we assume $F = F(X,t)$ is the continuous twice differentiable function. By applying Taylor's theorem, one has

$$\begin{aligned} dF(X,t) &= F_t dt + F_X dX + \frac{1}{2}F_{tt}(dt)^2 + \frac{1}{2}F_{xx}(dX)^2 + F_{tx}dtdX \\ &= F_t dt + F_X\big(a(X,t)dt + b(X,t)dW(t)\big) + \frac{1}{2}F_{tt}(dt)^2 \\ &+ \frac{1}{2}F_{XX}\big(a(X,t)dt + b(X,t)dW(t)\big)^2 \\ &+ F_{tX}dt\big(a(X,t)dt + b(X,t)dW(t)\big) \end{aligned} \tag{3.8}$$

By the definition of Wiener process, $dW(t)$ is a normal random variable with the mean of zero and the variance of Δt, namely

$$\begin{aligned} Var\big(dW(t)\big) &= E\Big(\big(dW(t)\big)^2\Big) - \Big(E\big(dW(t)\big)\Big)^2 \\ &= E\Big(\big(dW(t)\big)^2\Big) - 0 \\ &= \Delta t \\ &= dt \end{aligned} \tag{3.9}$$

Thus, we can approximate $\big(dW(t)\big)^2$ by Δt. In (3.8), if we ignore the term of $(dt)^2$ and $dtdW(t)$, by approximating $\big(dW(t)\big)^2 = dt$, we have

$$\begin{aligned} dF(X,t) &= F_t dt + a(X,t)F_X dt + b(X,t)F_X dW(t) + \frac{1}{2}b(X,t)^2F_{XX}dt \\ &= \big(a(X,t)F_X + F_t + \frac{1}{2}b(X,t)^2F_{XX}\big)dt + b(X,t)F_X dW(t), \end{aligned} \tag{3.10}$$

which is the same as (3.7) as we expect.

In the case that the portfolio is replicated by a risk-free bond $B(t)$ and a stock $S(t)$ at time t and this portfolio can be treated as the payoffs of the derivative $C(t)$, the following mathematical definitions hold. The risk-free bond, $B(t)$, can be defined by equation (3.2) since it is risk-free, namely

$$\frac{dB(t)}{B(t)} = rdt. \tag{3.11}$$

The stock, $S(t)$, can be defined by equation (3.5) since the stock has a stochastic movement,

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t). \tag{3.12}$$

The last term that we need to consider is the derivative payoff. The derivative price, $C(S,t)$, must depend on the other value, in our case, which is a stock, $S(t)$. From equation (3.7), one has

$$\begin{aligned} dC(S,t) = &\big(C(S,t)_t + \mu S(t)C(S,t)_S + \frac{1}{2}\sigma^2 S(t)^2 C(t)_{SS}\big)dt \\ &+ \sigma S(t)C(S,t)_S dW(t). \end{aligned} \tag{3.13}$$

There are many approaches to derive the Black-Scholes model. By the method of replicating the portfolio with self-financing assumption, such as replicating the derivative with a stock and a bond or replicating the bond with a stock and a derivative or replicating the stock with a bond and a derivative, we can derive the Black-Scholes equation [71].

3.3 Derivation from Asset Definition Perspective

From the previous section, a derivative is a financial security whose price is dependent upon or derived from the value of the other underlying assets, such as stocks, bonds, currencies, stock indexes and commodities. By this definition of derivative, it implies that the derivative depends on assets. If we assume that cash and real estate are the factors for the change of the derivative, the derivation of the Black-Scholes equation is the same as the previous method by replicating a portfolio by a stock and a bond.

$$\text{Asset} = \text{Cash} + \text{Real Estate} \tag{3.14}$$

The same concept of the replicating method can be applied. Cash is the most liquid asset and also is the most preferable physical form of currency as money. In economics, currency, as its name says, has fluctuation. It is crucial to make a note that currency fluctuation might be seen as both upward and downward movements. And this movement can be defined as a stochastic movement. Consequently, a stochastic term is mathematically assumed for cash definition. If we denote cash by F and assume that F follows the geometric Brownian motion as a part of Wiener process, then the stochastic process for F is

$$\frac{dF(t)}{F(t)} = \mu dt + \sigma dW(t) \tag{3.15}$$

which is the same process as the stock in (3.12) and μ is the mean rate of return of the cash.

The real estate, $E(t)$, is assumed to be the risk-free asset. The certain amount of return in the future can be expected. Mathematically, the real estate value, $E(t)$, can be described by following the equation (3.11), namely

$$\frac{dE(t)}{E(t)} = rdt. \tag{3.16}$$

If x is the unit number of cash or currency and y is the unit number of real estate. The value of the asset, $A(t)$, can be written as follows

$$A(t) = xF(t) + yE(t). \tag{3.17}$$

By self-financing assumption which implies that no money is added or withdrawn, the value of the asset changes due to the value change of the cash and the real estate, namely

$$dA(t) = xdF(t) + ydE(t). \tag{3.18}$$

Substituting (3.15) and (3.16) into (3.18), we have

$$\begin{aligned} dA(t) &= x\big(\mu F(t)dt + \sigma F(t)dW(t)\big) + y\big(rE(t)dt\big) \\ &= \big(x\mu F(t) + yrE(t)\big)dt + x\sigma F(t)dW(t) \end{aligned} \tag{3.19}$$

In order to mimic $C(t)$, we assume $A(t) = C(t)$ and $dA(t) = dC(t)$. And by the definition of derivative, a derivative value $C(t)$ depends on $F(t)$ and t. Then we have

$$dC(F(t),t) = \big(x\mu F(t) + yrE(t)\big)dt + x\sigma F(t)dW(t) \tag{3.20}$$

By Ito's lemma (3.7), we have

$$\begin{aligned} dC(F(t),t) = {} & \big(\mu F(t)C_F(F(t),t) + C_t(F(t),t) + \frac{1}{2}\sigma^2 F(t)^2 C_{FF}(F(t),t)\big)dt \\ & + \sigma F(t)C_F(F(t),t)dW(t) \end{aligned} \tag{3.21}$$

Therefore, from (3.20) and (3.21), we have the following equations

$$x\mu F(t) + yrE(t) = \mu F(t)C_F(F(t),t) + C_t(F(t),t) + \frac{1}{2}\sigma^2 F(t)^2 C_{FF}(F(t),t) \tag{3.22}$$

and

$$x\sigma F(t) = \sigma F(t)C_F(F(t),t). \tag{3.23}$$

From (3.23), we gain

$$x = C_F(F(t),t). \tag{3.24}$$

Substituting (3.24) into (3.17) yields

$$y = \frac{1}{E(t)}\big(C(F(t),t) - F(t)C_F(F(t),t)\big). \tag{3.25}$$

Substituting x and y from (3.24) and (3.25) into (3.22), we have

$$\begin{aligned} & C_F(F(t),t)\mu F(t) + \frac{1}{E(t)}\big(C(F(t),t) - F(t)C_F(F(t),t)\big)rE(t) \\ & = \mu F(t)C_F(F(t),t) + C_t(F(t),t) + \frac{1}{2}\sigma^2 F(t)^2 C_{FF}(F(t),t), \end{aligned} \tag{3.26}$$

that is

$$
\begin{aligned}
&C_F(F(t),t)\mu F(t) + Cr - F(t)rC_F(F(t),t) \\
&= \mu F(t)C_F(F(t),t) + C_t(F(t),t) + \frac{1}{2}\sigma^2 F(t)^2 C_{FF}(F(t),t) \qquad (3.27)
\end{aligned}
$$

By simplification, finally we obtain

$$
C_t(F(t),t) + rF(t)C_F(F(t),t) + \frac{1}{2}\sigma^2 F(t)^2 C_{FF}(F(t),t) = rC(F(t),t) \qquad (3.28)
$$

The above partial differential equation, (3.28), is known as Black-Scholes equation.

3.4 Derivation from n Stochastic Assets Perspective

In this section, we will generalize the idea of derivation of Black-Scholes equation from the same concept that we derive the model from the definition of derivative which states that a derivative is a financial security whose price is dependent upon or derived from the value of the other underlying assets, such as stocks, commodities, currencies, stock indexes and bonds. There are many kinds of underlying assets, but we assume that there are two main types of assets in mathematical aspect, asset with deterministic return, $E(t)$, and non-deterministic(stochastic) return, $F(t)$. We also assume that bond is the only one underlying asset defined in the category of the asset with deterministic return while the others follow the Wiener process. By the definition of derivative above, the derivative can be derived from various kinds of underlying assets. The derivation can be set as follows.

$$
C(F(t),t) = \big(x_1F_1(t) + x_2F_2(t) + ... + x_nF_n(t)\big) + yE(t) \qquad (3.29)
$$

where x_1,x_2...x_n and y are the number of each kind of assets. From (3.29), one has

$$dC(F(t),t) = \big(x_1 dF_1(t) + x_2 dF_2(t) + ... + x_n dF_n(t)\big) + ydE(t) \tag{3.30}$$

We mathematically assume that there are two types of assets. Bond is the deterministic asset and the others are stochastic assets. Thus, we have

$$\frac{dE(t)}{E(t)} = rdt, \tag{3.31}$$

$$\frac{dF_i(t)}{F_i(t)} = \mu_i dt + \sigma_i dW(t). \tag{3.32}$$

Substituting (3.31) and (3.32) into (3.30) gives

$$\begin{aligned} dC(F(t),t) = \Big(& x_1\big(\mu_1 F_1(t)dt + \sigma_1 F_1(t)dW_1(t)\big) + yrE(t)dt \\ & + x_2\big(\mu_2 F_2(t)dt + \sigma_2 F_2(t)dW_2(t)\big) + ... \\ & + x_n\big(\mu_n F_n(t)dt + \sigma_n F_n(t)dW_n(t)\big)\Big) \end{aligned} \tag{3.33}$$

Rearrange (3.33), we have

$$\begin{aligned} dC(F(t),t) = \Big(& \big(x_1\mu_1 F_1(t) + x_2\mu_2 F_2(t) + ... + x_n\mu_n F_n(t)\big)dt + yrE(t)dt \\ & + \big(x_1\sigma_1 F_1(t)dW_1(t) + x_2\sigma_2 F_2(t)dW_2(t) + ... + x_n\sigma_n F_n(t)dW_n(t)\big). \end{aligned} \tag{3.34}$$

By Ito's lemma, the derivative can be derived as

$$\begin{aligned} dC(F(t),t) = & \sum_{i=1}^{n}\Bigg[\big(\mu_i F_i(t) C_{F_i}(F(t),t)\Bigg] dt + C_t(F(t),t)dt \\ & + \sum_{i=1}^{n}\Bigg[\frac{1}{2}\sigma_i^2 F_i(t)^2 C_{F_iF_i}(F(t),t)\big)dt\Bigg] \\ & + \sum_{i=1}^{n}\Bigg[\sigma_i F_i(t) C_{F_i}(F(t),t)dW_i(t)\Bigg]. \end{aligned} \tag{3.35}$$

From (3.34) and (3.35), by the same method as in previous section, we have

$$\begin{aligned}&\big(x_1F_1(t)\mu_1 + x_2F_2(t)\mu_2 + ... + x_nF_n(t)\mu_n\big) + yE(t)r\\&= \sum_{i=1}^{n}\left[\mu_iF_i(t)C_{F_i}(F(t),t)\right] + C_t(F(t),t) + \sum_{i=1}^{n}\left[\frac{1}{2}\sigma_i^2F_i(t)^2C_{F_iF_i}(F(t),t)\right],\end{aligned} \tag{3.36}$$

$$\begin{aligned}&x_1\sigma_1F_1(t)dW_1(t) + x_2\sigma_2F_2(t)dW_2(t) + ... + x_n\sigma_nF_n(t)dW_n(t)\\&= \sum_{i=1}^{n}\left[\sigma_iF_i(t)C_{F_i}(F(t),t)dW_i(t)\right].\end{aligned} \tag{3.37}$$

From (3.37), we obtain

$$x_i = C_{F_i}(F(t),t). \tag{3.38}$$

Substituting the above equation (3.38) into (3.29) yields

$$C(F(t),t) = \sum_{i=1}^{n}\left[F_i(t)C_{F_i}(F(t),t)\right] + yE(t), \tag{3.39}$$

that is

$$C(F(t),t) - \sum_{i=1}^{n}\left[F_i(t)C_{F_i}(F(t),t)\right] = yE(t). \tag{3.40}$$

Substituting (3.38) and (3.40) into (3.36) yields

$$\begin{aligned}&\sum_{i=1}^{n}\left[C_{F_i}(F(t),t)\mu_iF_i(t)\right] + rC(F(t),t) - \sum_{i=1}^{n}\left[rF_i(t)C_{F_i}(F(t),t)\right]\\&= \sum_{i=1}^{n}\left[\mu_iF_i(t)C_{F_i}(F(t),t))\right] + C_t(F(t),t) + \sum_{i=1}^{n}\left[\frac{1}{2}\sigma_i^2F_i(t)^2C_{F_iF_i}(F(t),t))\right].\end{aligned} \tag{3.41}$$

By simplification, finally we obtains

$$\begin{aligned}&C_t(F(t),t) + \sum_{i=1}^{n}\left[rF_i(t)C_{F_i}(F(t),t)\right] + \sum_{i=1}^{n}\left[\frac{1}{2}\sigma_i^2F_i(t)^2C_{F_iF_i}(F(t),t)\right]\\&= rC(F(t),t).\end{aligned} \tag{3.42}$$

The result can be explained as an n-dimensional Black-Scholes equation.

3.5 Derivation from n Stochastic Assets with Fixed-income Perspective

In the Black-Scholes-Merton model [4, 5], Robert Merton extended the option pricing model by considering the dividend parameter. The model assumes a constant dividend payment. In this section we will derive the Black-Scholes-Merton equation for the case with n stochastic assets with fixed-income perspective. If the asset i is paying the dividend, at time dt the underlying asset pays out a dividend $D_iF_i(t)dt$, where D_i is a fixed income rate or a constant dividend yield of asset i. We can derive the model of asset i as

$$dF_i(t) = (\mu_i + D_i)F_i(t)dt + \sigma_i F_i(t)dW_i(t) \tag{3.43}$$

The same approach of derivation as for the previous sections can be applied. As before, the asset value is

$$C(F(t), t) = \big(x_1F_1(t) + x_2F_2(t) + ... + x_nF_n(t)\big) + yE(t) \tag{3.44}$$

where x_1,x_2...x_n and y are the number of each of the assets. The increment of the asset value is

$$dC(F(t), t) = \big(x_1dF_1(t) + x_2dF_2(t) + ... + x_ndF_n(t)\big) + ydE(t) \tag{3.45}$$

Again, by the assumption of two types of assets, deterministic and stochastic assets, we let

$$dE(t) = rE(t)dt \tag{3.46}$$

$$dF_i(t) = (\mu_i + D_i)F_i(t)dt + \sigma_i F_i(t)dW_i(t) \tag{3.47}$$

Substituting (3.46) and (3.47) into (3.45) yields

$$\begin{aligned}
dC(F(t),t) = \Big(& x_1\big((\mu_1 + D_1)F_1(t)dt + \sigma_1 F_1(t)dW_1(t)\big) \\
& + x_2\big((\mu_2 + D_2)F_2(t)dt + \sigma_2 F_2(t)dW_2(t)\big) \\
& + ... \\
& + x_n\big((\mu_n + D_n)F_n(t)dt + \sigma_n F_n(t)dW_n(t)\big)\Big) \\
& + yrE(t).
\end{aligned} \tag{3.48}$$

Rearranging (3.46) gives

$$\begin{aligned}
dC(F(t),t) = \Big(& \big(x_1(\mu_1 + D_1)F_1(t) + x_2(\mu_2 + D_2)F_2(t) + ... + x_n(\mu_n + D_n)F_n(t)\big)dt \\
& + yrE(t)dt \\
& + \big(x_1\sigma_1 F_1(t)dW_1(t) + x_2\sigma_2 F_2(t)dW_2(t) + ... + x_n\sigma_n F_n(t)dW_n(t)\big).
\end{aligned} \tag{3.49}$$

By Ito's lemma, the derivative can be calculated by

$$\begin{aligned}
dC(F(t),t) = & \sum_{i=1}^{n} \Bigg[\mu_i F_i(t) C_{F_i}(F(t),t)\Bigg] dt + C_t(F(t),t)dt \\
& + \sum_{i=1}^{n} \Bigg[\frac{1}{2}\sigma_i^2 F_i(t)^2 C_{F_iF_i}(F(t),t)\big)dt\Bigg] \\
& + \sum_{i=1}^{n} \Bigg[\sigma_i F_i(t) C_{F_i}(F(t),t) dW_i(t)\Bigg].
\end{aligned} \tag{3.50}$$

From (3.49) and (3.50), we obtain

$$\begin{aligned}
& x_1(\mu_1 + D_1)F_1(t) + x_2(\mu_2 + D_2)F_2(t) + ... + x_n(\mu_n + D_n)F_n(t) + ryE(t) \\
& = \sum_{i=1}^{n} \Bigg[\mu_i F_i(t) C_{F_i}(F(t),t)\Bigg] + C_t(F(t),t) + \sum_{i=1}^{n} \Bigg[\frac{1}{2}\sigma_i^2 F_i(t)^2 C_{F_iF_i}(F(t),t)\Bigg],
\end{aligned} \tag{3.51}$$

$$\begin{aligned}
& x_1\sigma_1 F_1(t)dW_1(t) + x_2\sigma_2 F_2(t)dW_2(t) + ... + x_n\sigma_n F_n(t)dW_n(t) \\
& = \sum_{i=1}^{n} \Bigg[\sigma_i F_i(t) C_{F_i}(F(t),t) dW_i(t)\Bigg].
\end{aligned} \tag{3.52}$$

From (3.52), we have

$$x_i = C_{F_i}(F(t), t) \tag{3.53}$$

Substitute (3.53) into (3.44), then (3.44) becomes

$$C(F(t), t) = \sum_{i=1}^{n} \Big[F_i(t) C_{F_i}(F(t), t) \Big] + yE(t), \tag{3.54}$$

that is

$$C(F(t), t) - \sum_{i=1}^{n} \Big[F_i(t) C_{F_i}(F(t), t) \Big] = yE(t). \tag{3.55}$$

Substituting (3.55) and (3.53) into (3.51) gives

$$\begin{aligned}
&\sum_{i=1}^{n} \Big[C_{F_i}(F(t), t)(\mu_i + D_i) F_i(t) \Big] + C(F(t), t) r - \sum_{i=1}^{n} \Big[r F_i(t) C_{F_i}(F(t), t) \Big] \\
&= \sum_{i=1}^{n} \Big[\mu_i F_i(t) C_{F_i}(F(t), t) \Big] + C_t(F(t), t) + \sum_{i=1}^{n} \Big[\frac{1}{2} \sigma_i^2 F_i(t)^2 C_{F_i F_i}(F(t), t) \Big]
\end{aligned} \tag{3.56}$$

By simplification, we finally obtain

$$\begin{aligned}
&C_t(F(t), t) + \sum_{i=1}^{n} \Big[(r - D) F_i(t) C_{F_i}(F(t), t) \Big] + \frac{1}{2} \sum_{i=1}^{n} \Big[\sigma_i^2 F_i(t)^2 C_{F_i F_i}(F(t), t) \Big] \\
&= r C(F(t), t)
\end{aligned} \tag{3.57}$$

which is the n-dimensional Black-Scholes-Merton model for a European Call Option pricing with the constant dividend.

3.6 Concluding Remarks

In this chapter, we examine the Black-Scholes-Merton framework. In order to attain the extension of the option pricing model, we need to examine the framework behind the derivation of the classic Black-Scholes model. This chapter brings out the alternative perspective to derive the Black-Scholes model. By considering the definition of the derivative, we define assets mathematically and we can derive the classic option pricing. By this method, the derivation is similar to the derivation of derivative by replicating the derivative with stock and bond. Also the problem of n kinds of stochastic assets leads to derivation of the n-dimensional Black-Scholes type equations which extend the classical model for more complex applications. And the last section is to derive the model by considering the fixed return of the asset or constant dividend to derive the extended classic option formula, the n-dimensional Black-Scholes-Merton type model. In this section, the idea to prove the model especially the n assets problem is the important key which will play the significant role in Chapter 5 for the Greek parameter propositions.

Chapter 4

Option Pricing Model with Stochastic Earning Yield

4.1 General

In finance, an option is a derivative financial instrument that traders consider for their investment in order to gain the confidence to make a profit in the stock market. For decades, in the Black-Scholes-Merton structure, the option price $C(t)$ is determined by the well-known factors including stock price $S(t)$, strike price of an option $K(t)$, dividend yield $\delta(t)$, volatility σ, interest rate r , time t and maturity T [3–5]. To make the option pricing model more accurate under different finance situation in the real world, stochastic parameters are taken into the option valuation model. Dividend yield is one of the most important factors to be determined as stochastic. Very few work has been done to explicitly consider stochastic dividend yield [7–9,12,38–40,43]. In the evolution of Stochastic process, an Ornstein-Uhlenbeck process has regulated the mathematical model to valuate the option price [38, 39, 48].

Economically, not only dividend yield but also P/E ratio is the most general data source which can be acquired from everyday newspaper. Consequently, in this chapter, the Black-Scholes-Merton framework is extended by considering the P/E ratio, a reciprocal of earning yield $\xi(t)$. To overcome the problem of how to take account either P/E ratio or earning yield, in this chapter, the relationship of P/E ratio with the option pricing model is determined. Earning yield is significantly playing an important role to the model of dividend yield. A generalization of the Ornstein-Uhlenbeck process is defined and named as GOU process. Stochastic differential equations are then set up to explain the situation of stock value by incorporating with stochastic dividend yield. The GOU process is applied to describe the relationship between stochastic earning yield and stochastic dividend yield. Inspired by Abraham Lioui approach [38], we define a Weiner Process with n risk factors and derive the explicit formulas for the European call and put options taking into account the stochastic earning yield.

The purpose of this chapter is to study and develop European option pricing models in which dividend yield is taken into account by earning yield, a reciprocal of P/E ratio, for every $t \in [0, T]$. The Ornstein-Uhlenbeck process is generalized and defined as GOU process. Based on the Black-Scholes-Merton model structure, the firm's stock follows the geometric Brownian motion process. Dividend yield is assumed to follow the GOU process under a stochastic mean-reverting earning yield. The existence of three stochastic components $dW_i(t)$ of the dynamic GOU allows the firm's dividend to randomly deviate from the earning yield flow. Stock price, dividend yield and earning yield are modeled under a stochastic mean-reverting market price of risk (MPR). Explicit formulas are derived for the European call and put option prices.

After we propose the new option pricing model by considering the earning yield factor in this chapter, we will investigate the characteristics and properties of the model and also we will apply the model to test the possibility of real-world application by running the empirical test to the model in Chapter 6.

4.2 The GOU Process

In this work, we propose the new stochastic process, named GOU Process, as the generalized process of the Ornstein-Unlenbeck process. Ornstein-Uhlenbeck defined a modification of a random walk process, Weiner process, in continuous time for which the characteristics of the walk has the movement to move back to the center. The process can be considered to be a modification of the random walk in continuous time, or Wiener process, in which the properties of the process have been changed so that there is a tendency of the walk to move back towards a central location, with a greater attraction when the process is further away from the center.

In recent years, the Ornstein-Uhlenbeck process has been applied to the option valuation model and has been playing an outstanding role as a tool for pricing the models with stochastic parameters. [9, 38, 39, 61]. This model is widely used for explaining the stochastic dynamics of the parameter, such as volatility, interest rate, commodity prices and currency exchange rate [73].

Definition 4.1. *The Ornstein-Uhlenbeck process is a stochastic process, $x(t)$, which satisfies the following stochastic differential equation:*

$$dx(t) = \kappa(\mu - x(t))dt + \sigma dW(t)) \tag{4.1}$$

where $W(t)$ denotes the wiener process, a standard Brownian motion, on $t \in [0, \infty)$ and $\kappa > 0$, $\sigma > 0$ and μ are constant parameters.

Since there exists such a situation of a parameter which depends on itself or other parameters or both, it is assumed that there exists such a stochastic process which follows the normal Ornstein-Uhlenbeck process with the dependence of not only its own process, but also other processes. Roughly speaking, this process explicate an idea of, for instance, the velocity of a massive Brownian particle under the influence of many kinds of frictions. In this case, a process, $X(t)$, named as a GOU process, is a generalization of the Ornstein-Uhlenbeck process, and we define the process as follows.

Definition 4.2. *The generalization of the Ornstein-Uhlenbeck process, named as GOU process, is a stochastic process, $X(t)$, which satisfies the following stochastic differential equation:*

$$\begin{aligned} dX(t) = {} & \kappa_0(\mu_0 - X(t))dt \\ & + \kappa_1(\mu_1 - Z_1(t))dt + \kappa_2(\mu_2 - Z_2(t))dt + ... + \kappa_n(\mu_n - Z_n(t))dt \\ & + \sigma_0 dW_0(t) + \sigma_1 dW_1(t) + \sigma_2 dW_2(t) + ... + \sigma_n dW_n(t)) \end{aligned} \tag{4.2}$$

where $Z_i(t)$ denotes the GOU process, $W(t)$ denotes the Wiener process, on $t \in [0, \infty)$ and $\kappa_0 > 0$, $\sigma_0 > 0$,$\kappa_i \geq 0$, $\sigma_i \geq 0$ and $\mu_i \in \mathbb{R}$ are constant parameters where $i = 1, 2, ..., n$.

Under the martingale measure Q, the risk-neutral probability, we assume that within the movement of underlying assets, the risky constituents of market movements can affect a standard Brownian motion. The risky constituents are the risk factors which can effect the stated underlying asset. For example, the market price of risk makes an impact on the stock price. In aspect of this situation, when each risky element within a Weiner process is followed by a GOU process (see Definition 4.2), we define this stochastic process as follows.

Definition 4.3. *The GOU Weiner process, is a continuous-time stochastic process, $\widehat{W}(t)$, defined by*

$$\widehat{W}(t) = W(t) + \int_0^t \kappa_1(s)ds + \int_0^t \kappa_2(s)ds + ... + \int_0^t \kappa_n(s)ds \tag{4.3}$$

where $W(t)$ denotes the wiener process and $\kappa_i(t)$ is the GOU process when $i = 1, 2, ..., n$.

The purpose of this chapter is to construct a model of option pricing based on the actual financial circumstance. All stochastic processes and necessary conditions are taken into account in the mathematical model. Hence, a general GOU process is defined for the advantage of constructing financial models for the general case.

Definition 4.4. *If $X(t)$ is a GOU process, the general GOU process is a stochastic process , $\widehat{X}(t)$, defined by the following stochastic differential equation:*

$$\begin{aligned} d\widehat{X}(t) &= \kappa_0(\mu_0 - X(t))dt \\ &\quad + \kappa_1(\mu_1 - \widehat{Z}_1(t))dt + \kappa_2(\mu_2 - \widehat{Z}_2(t))dt + ... + \kappa_n(\mu_n - \widehat{Z}_n(t))dt \\ &\quad + \sigma_0 d\widehat{W}_0(t) + \sigma_1 d\widehat{W}_1(t) + \sigma_2 d\widehat{W}_2(t) + ... + \sigma_n d\widehat{W}_n(t)) \end{aligned} \tag{4.4}$$

where $\widehat{Z}_i(t)$ denotes the general GOU process, $\widehat{W}(t)$ denotes the GOU Wiener process on $t \in [0, \infty)$, and $\kappa_0 > 0$, $\sigma_0 > 0$,$\kappa_i \geq 0$, $\sigma_i \geq 0$ and $\mu_i \in \mathbb{R}$ are all constant parameters where $i = 1, 2, ..., n$.

By the definitions above, the extended pricing model taking into account the P/E ratio or earning yield can be constructed mathematically, and is to be presented in the next section.

4.3 The Option Pricing Model Setting

To establish the extended pricing models, the financial market is assumed to be complete and no arbitrage. Under the probability space $(\Omega, \mathbb{P}, \mathbb{F})$, Ω is the pricing outcomes space, $\mathbb{F}$ is the σ-algebra denoting measurable events, and $\mathbb{P}$ is the probability measure. There exists a fixed martingale measure $\mathbb{Q}$ which presumably equals to the probability measure $\mathbb{P}$ such that the asset price, discounted at the risk-free interest rate, and cumulated discounted dividends are martingales. This assumption guarantees that the market has no arbitrage opportunity [74]. All stochastic processes in such pricing environment are adapted to the filtration, $\{\mathcal{F}_t\}$, generated by the Wiener processes. According to Girsanov's theorem with multiple Brownian motions, there exist $\mathcal{F}_t$-adapted processes $\kappa_1(t)$, $\kappa_2(t)$ and $\kappa_3(t)$. The equivalent martingale measure $\mathbb{Q}$ and the measurable probability $\mathbb{P}$ are related by the following Radon-Nikodym derivative equation [75]

$$\begin{aligned}\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\Big(&- \int_0^t \kappa_1(s)dW_1(s) - \int_0^t \kappa_2(s)dW_2(s) - \int_0^t \kappa_3(s)dW_3(s) \\ &- \frac{1}{2}\int_0^t \Big(\kappa_1(s)^2 + \kappa_2(s)^2 + \kappa_3(s)^2\Big)ds\Big) \qquad , \mathbb{P} - a.s. \end{aligned} \tag{4.5}$$

The above equation is defined on a complete probability space $(\Omega, \mathbb{P}, \mathbb{F})$. Brownian motions, $dW_1(s)$, $dW_2(s)$ and $dW_3(s)$, are assumed to be one-dimensional uncorrelated Weiner Process.

The risky asset price corresponds to a stock process $(S(t))$ in such a way that, in an infinitesimal amount of time dt, the infinitesimal amount of stock price $dS(t)$ has mean $((\mu - \delta)S(t)dt$ (when $\mu = \mu(t, S(t), \delta(t))$ and $\delta = \delta(t)$). The stock price can be modeled by the following stochastic differential equation (SDE)

$$\frac{dS(t)}{S(t)} = \Big(\mu_S(t, S(t), \delta(t)) - \delta(t)\Big)dt + \sigma_S dW_1(t), \tag{4.6}$$

where $S(0) > 0, \mu_S(t, S(t), \delta(t))$ is the total stock yield and σ_S is a positive volatility.

In this work, the P/E ratio is to be considered in the option pricing model. By the principle of finance, dividend yield depends on the P/E ratio by

$$\begin{aligned}\delta(t) &= \frac{1}{P/E}.\Big(\frac{\text{Dividend}}{\text{Earning Per Share}}\Big)\\ &= (\text{Earning Yield}).\Big(\frac{\text{Dividend}}{\text{Earning Per Share}}\Big).\end{aligned} \tag{4.7}$$

We assume that the derivative of dividend yield depends on the earning yield, a reciprocal of the P/E ratio.

Consequently, dividend yield, $\delta(t)$, can be assumed to follow the GOU process. The model is proposed as follows.

$$\begin{aligned}d\delta(t) &= \theta_\delta\Big(\mu_\delta - \delta(t)\Big)dt + \theta_\xi\Big(\mu_\xi - \xi(t)\Big)dt\\ &\quad + \Big(\sigma_{\delta 1}dW_1(t) + \sigma_{\delta 2}dW_2(t) + \sigma_{\delta 3}dW_3(t)\Big),\end{aligned} \tag{4.8}$$

where $\delta(0) = 0$, θ_δ, μ_δ, θ_ξ, μ_ξ, and $\sigma_{\delta i}$ are constants where $i = 1, 2, 3$

Earning yield, $\xi(t)$, is defined by the following Ornstein-Uhlenbeck process with the random fluctuations of three parameters (as of the GOU process)

$$d\xi(t) = \theta_\xi\Big(\mu_\xi - \xi(t)\Big)dt + \Big(\sigma_{\xi 1}dW_1(t) + \sigma_{\xi 2}dW_2(t) + \sigma_{\xi 3}dW_3(t)\Big), \tag{4.9}$$

where $\xi(0) = 0$, θ_ξ, μ_ξ, θ_ξ, μ_ξ, and $\sigma_{\xi i}$ are constants when $i = 1, 2, 3$.

Note that the equation (4.9) is correlated to (4.8), and (4.8) is correlated to (4.6). These give the relationship among corresponding parameters: $S(t)$, $\delta(t)$ and $\xi(t)$. In particular case, the absence of correlation between the two processes (or among three processes) can be derived by setting the parameters θ_ξ, $\sigma_{\delta i}$ and/or $\sigma_{\xi i}$ to zero.

The market price of risk is considered as a risk factor κ_i by definition 4.3 and is assumed to follow the GOU process as follows

$$d\kappa_i(t) = \theta_{\kappa i}\Big(\mu_{\kappa i} - \kappa_i(t)\Big)dt + \sigma_{\kappa i} dW_i(t), \tag{4.10}$$

where $\kappa(0) = 0$, θ_{κ_i}, μ_{κ_i}, and σ_{κ_i} are positive constants when $i = 1, 2, 3$.

The models of stock (4.6), dividend (4.8), earning yiled (4.9) and market price of risk (4.10) are affected by the market risks; as a result, in the next section, the pricing models are constructed to capsulate the risk factors, which yield the advantages for asset pricing.

4.4 The Method and Proof of Pricing Model

With the models setting from the previous section and by analytic approach of Abraham Lioui [38], the following proofs are derived. The formulas of European options are proved analytically with the assumptions of arbitrage-free and frictionless market.

When the market price of risk (MPR), $\kappa_i(t)$, is considered, the kernel of movement is changed by the market risk. As a result, the Wiener processes $W_i(t)$ which contains the movement of the market in the models (4.6), (4.8), (4.9) and (4.10) are replaced by the GOU Wiener process with MPR, $\widehat{W}_i(t)$, instead. Since one parameter of risk movement, MPR, is determined and by using the definition 4.3, the GOU Wiener process is derived as follows

$$\widehat{W}_i(t) = W_i(t) + \int_0^t \kappa_i(s) ds. \tag{4.11}$$

The dynamic models (4.6), (4.8), (4.9) and (4.10) are changed. The equation of the stock price (4.6) becomes

$$\frac{dS(t)}{S(t)} = \Big(\mu_S(t, S(t), \delta(t)) - \delta(t)\Big)dt + \sigma_S d\widehat{W}_1(t). \tag{4.12}$$

By setting $\mu_S(t, S(t), \delta(t)) = r$, the rate of stock return, we have

$$\frac{dS(t)}{S(t)} = \Big(r - \delta(t)\Big)dt + \sigma_S d\widehat{W}_1(t). \tag{4.13}$$

Because of the fact that the MPR have an influence on not only the directional movement of the underlying asset, but also the mean of the asset value directly. The intrinsic value of the dividend yield can be defined in term of earning yield by our assumption and can be described by the following processes

$$
\begin{aligned}
d\delta(t) = {} & \theta_\delta\Big(\mu_\delta - \frac{1}{\theta_\delta}\Big(\sigma_{\delta 1}\kappa_1(t) + \sigma_{\delta 2}\kappa_2(t) + \sigma_{\delta 3}\kappa_3(t)\Big) - \delta(t)\Big)dt \\
& + \theta_\xi\Big(\mu_\xi - \xi(t)\Big)dt \\
& + \Big(\sigma_{\delta 1}d\widehat{W}_1(t) + \sigma_{\delta 2}d\widehat{W}_2(t) + \sigma_{\delta 3}d\widehat{W}_3(t)\Big), \qquad (4.14)
\end{aligned}
$$

where

$$
\begin{aligned}
d\xi(t) = {} & \theta_\xi\Big(\mu_\xi - \frac{1}{\theta_\xi}\Big(\sigma_{\xi 1}\kappa_1(t) + \sigma_{\xi 2}\kappa_2(t) + \sigma_{\xi 3}\kappa_3(t)\Big) - \xi(t)\Big)dt \\
& + \Big(\sigma_{\xi 1}d\widehat{W}_1(t) + \sigma_{\xi 2}d\widehat{W}_2(t) + \sigma_{\xi 3}d\widehat{W}_3(t)\Big), \qquad (4.15)
\end{aligned}
$$

in which the equation (4.10) is governed by the risk of market movement. The market price of risk, MPR, is defined by the Ornstein-Uhlenbeck process as

$$
d\kappa_i(t) = \bar{\theta}_{\kappa i}\Big(\bar{\mu}_{\kappa i} - \kappa_i(t)\Big)dt + \sigma_{\kappa i}d\widehat{W}_i(t), \qquad (4.16)
$$

where

$$
\begin{aligned}
\bar{\theta}_{\kappa i} &= \theta_{\kappa i}(1 + \sigma_{\kappa i}), \\
\bar{\mu}_{\kappa i} &= \frac{\mu_{\kappa i}}{1 + \sigma_{\kappa i}}.
\end{aligned}
$$

In order to derive the option pricing formula, European call option must be derived before obtaining European put option formula by using the concept of put-call parity. The European call option, $C(t)$, is defined as [38, 39]

$$
\begin{aligned}
C(t) &= E^Q\Big[e^{-r(T-t)}[S(T) - K]^+|F_t\Big] \qquad (4.17) \\
&= e^{-r(T-t)}E^Q\Big[S(T)\mathbf{1}_{S(T)>K}|F_t\Big] - Ke^{-r(T-t)}E^Q\Big[\mathbf{1}_{S(T)>K}|F_t\Big] \qquad (4.18)
\end{aligned}
$$

where

$$\mathbf{1}_{S(T)>K} = \begin{cases} 1 & \text{if} \quad S(T) > K, \\ 0 & \text{if} \quad \text{otherwise.} \end{cases}$$

E^Q refers to the mean value of the quantity under the risk-neutral probability measure $\mathbb{Q}$ as defined previously in section 4.3, and F_t refers to the filtration generated by the Wiener process defined perviously.

The distribution of the stock price $S(T)$, which relies on the factors of volatility, dividend yield and Brownian movement is obtained by solving (4.13)

$$S(T) = S(t) \exp\Big((r - \frac{\sigma_S^2}{2})(T-t) - \int_t^T \delta(s)ds + \sigma_S \int_t^T d\widehat{W}_1(S)\Big). \tag{4.19}$$

By observing the equation (4.19), the first term that we need to determine is the dividend term, $\int_t^T \delta(s)ds$. This term can be described as the total dividend yield in the future from time t to maturity T, and can be derived from (4.14).

Rearranging (4.14) yields

$$\begin{aligned} \frac{1}{\theta_\delta} d\delta(t) = &\Big(\mu_\delta - \delta(t)\Big)dt + \frac{\theta_\xi}{\theta_\delta}\Big(\mu_\xi - \xi(t)\Big)dt \\ &- \frac{1}{\theta_\delta}\Big(\sigma_{\delta 1}\kappa_1(t) + \sigma_{\delta 2}\kappa_2(t) + \sigma_{\delta 3}\kappa_3(t)\Big)dt \\ &+ \frac{1}{\theta_\delta}\Big(\sigma_{\delta 1} d\widehat{W}_1(t) + \sigma_{\delta 2} d\widehat{W}_2(t) + \sigma_{\delta 3} d\widehat{W}_3(t)\Big), \end{aligned} \tag{4.20}$$

that is

$$\begin{aligned} \delta(t)dt = &\mu_\delta dt - \frac{1}{\theta_\delta} d\delta(t) + \frac{\theta_\xi}{\theta_\delta}\mu_\xi dt - \frac{\theta_\xi}{\theta_\delta}\xi(t)dt \\ &- \frac{1}{\theta_\delta}\Big(\sigma_{\delta 1}\kappa_1(t) + \sigma_{\delta 2}\kappa_2(t) + \sigma_{\delta 3}\kappa_3(t)\Big)dt \\ &+ \frac{\sigma_{\delta 1}}{\theta_\delta} d\widehat{W}_1(t) + \frac{\sigma_{\delta 2}}{\theta_\delta} d\widehat{W}_2(t) + \frac{\sigma_{\delta 3}}{\theta_\delta} d\widehat{W}_3(t). \end{aligned} \tag{4.21}$$

By integrating both side of equation (4.21), we have

$$\int_t^T \delta(s)ds = \mu_\delta(T-t) - \frac{1}{\theta_\delta}\Big(\delta(T)-\delta(t)\Big) + \frac{\theta_\xi\mu_\xi}{\theta_\delta}(T-t)$$
$$- \frac{\theta_\xi}{\theta_\delta}\int_t^T \xi(s)ds - \frac{1}{\theta_\delta}\int_t^T \Big(\sigma_{\delta1}\kappa_1(s) + \sigma_{\delta2}\kappa_2(s) + \sigma_{\delta3}\kappa_3(s)\Big)ds$$
$$+ \frac{\sigma_{\delta1}}{\theta_\delta}\int_t^T d\widehat{W}_1(s) + \frac{\sigma_{\delta2}}{\theta_\delta}\int_t^T d\widehat{W}_2(s) + \frac{\sigma_{\delta3}}{\theta_\delta}\int_t^T d\widehat{W}_3(s). \tag{4.22}$$

To solve this problem, we compare the above equation (4.22) to the main equation of the distribution of stock price (4.19). As a result we can see that the dividend function at maturity T, $\delta(T)$, the integral term $\int_t^T \xi(s)ds$ and $\int_t^T \Big(\sigma_{\delta1}\kappa_1(s) + \sigma_{\delta2}\kappa_2(s) + \sigma_{\delta3}\kappa_3(s)\Big)ds$ must be solved.

We consider the MPR term first, $\int_t^T \Big(\sigma_{\delta1}\kappa_1(s) + \sigma_{\delta2}\kappa_2(s) + \sigma_{\delta3}\kappa_3(s)\Big)ds$. From (4.16), we derive

$$\int_t^T \kappa_i(s)ds = (\bar{\mu}_{\kappa i}(T-t) - \frac{1}{\bar{\theta}_{\kappa i}}\Big(\kappa_i(T) - \kappa_i(t)\Big) + \frac{\sigma_{\kappa i}}{\bar{\theta}_{\kappa i}}\int_t^T d\widehat{W}_i(s). \tag{4.23}$$

By solving the exact solution from (4.23), we have

$$\kappa_i(T) = \kappa_i(t)e^{-\bar{\theta}_{\kappa i}(T-t)} + \bar{\mu}_{\kappa i}(1 - e^{-\bar{\theta}_{\kappa i}(T-t)}) + \sigma_{\kappa i}\int_t^T e^{-\bar{\theta}_{\kappa i}(T-s)}d\widehat{W}_i(s). \tag{4.24}$$

Thus, (4.23) becomes

$$\int_t^T \kappa_i(s)ds = \frac{1}{\bar{\theta}_{\kappa i}}\kappa_i(t)\Big(1 - e^{-\bar{\theta}_{\kappa i}(T-t)}\Big)$$
$$+ \bar{\mu}_{\kappa i}(T-t) - \frac{1}{\bar{\theta}_{\kappa i}}\bar{\mu}_{\kappa i}\Big(1 - e^{-\bar{\theta}_{\kappa i}(T-t)}\Big)$$
$$+ \int_t^T \Big(\frac{\sigma_{\kappa i}}{\bar{\theta}_{\kappa i}} - \frac{\sigma_{\kappa i}}{\bar{\theta}_{\kappa i}}e^{-\bar{\theta}_{\kappa i}(T-s)}\Big)d\widehat{W}_i(s). \tag{4.25}$$

Substituting the above solution (4.25) into (4.22), we obtain

$$\begin{aligned}
\int_t^T \delta(s)ds = \mu_\delta(T-t) &- \frac{1}{\theta_\delta}\Big(\delta(T)-\delta(t)\Big) + \frac{\theta_\xi\mu_\xi}{\theta_\delta}(T-t) - \frac{\theta_\xi}{\theta_\delta}\int_t^T \xi(s)ds \\
&- \frac{\sigma_{\delta 1}\bar{\mu}_{\kappa 1}}{\theta_\delta}\left((T-t)-\frac{1}{\bar{\theta}_{\kappa 1}}\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right) \\
&- \frac{\sigma_{\delta 2}\bar{\mu}_{\kappa 2}}{\theta_\delta}\left((T-t)-\frac{1}{\bar{\theta}_{\kappa 2}}\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right) \\
&- \frac{\sigma_{\delta 3}\bar{\mu}_{\kappa 3}}{\theta_\delta}\left((T-t)-\frac{1}{\bar{\theta}_{\kappa 3}}\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right) \\
&- \frac{\sigma_{\delta 1}}{\theta_\delta\bar{\theta}_{\kappa 1}}\kappa_1(t)\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big) \\
&- \frac{\sigma_{\delta 2}}{\theta_\delta\bar{\theta}_{\kappa 2}}\kappa_2(t)\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) \\
&- \frac{\sigma_{\delta 3}}{\theta_\delta\bar{\theta}_{\kappa 3}}\kappa_3(t)\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big) \\
&+ \frac{\sigma_{\delta 1}}{\theta_\delta}\int_t^T\Big(1-\frac{\sigma_{\kappa 1}}{\bar{\theta}_{\kappa 1}}+\frac{\sigma_{\kappa 1}}{\bar{\theta}_{\kappa 1}}e^{-\bar{\theta}_{\kappa 1}(T-s)}\Big)d\widehat{W_1}(s) \\
&+ \frac{\sigma_{\delta 2}}{\theta_\delta}\int_t^T\Big(1-\frac{\sigma_{\kappa 2}}{\bar{\theta}_{\kappa 2}}+\frac{\sigma_{\kappa 2}}{\bar{\theta}_{\kappa 2}}e^{-\bar{\theta}_{\kappa 2}(T-s)}\Big)d\widehat{W_2}(s) \\
&+ \frac{\sigma_{\delta 3}}{\theta_\delta}\int_t^T\Big(1-\frac{\sigma_{\kappa 3}}{\bar{\theta}_{\kappa 3}}+\frac{\sigma_{\kappa 3}}{\bar{\theta}_{\kappa 3}}e^{-\bar{\theta}_{\kappa 3}(T-s)}\Big)d\widehat{W_3}(s). \qquad (4.26)
\end{aligned}$$

Now we will solve $\delta(T)$ from (4.14) by solving the exact solution with integrating factor. The solution can be derived as follows.

$$\begin{aligned}
\delta(T) = \delta(t)e^{-\theta_\delta(T-t)} &+ \mu_\delta\Big(1-e^{-\theta_\delta(T-t)}\Big) \\
&+ \frac{\theta_\xi\mu_\xi}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big) - \theta_\xi\int_t^T e^{-}\theta_\delta(T-s)\xi(s)ds \\
&- \sigma_{\delta 1}\int_t^T e^{-\theta_\delta(T-s)}\kappa_1(s)ds \\
&- \sigma_{\delta 2}\int_t^T e^{-\theta_\delta(T-s)}\kappa_2(s)ds \\
&- \sigma_{\delta 3}\int_t^T e^{-\theta_\delta(T-s)}\kappa_3(s)ds \\
&+ \sigma_{\delta 1}\int_t^T e^{-\theta_\delta(T-s)}d\widehat{W_1}(s) \\
&+ \sigma_{\delta 2}\int_t^T e^{-\theta_\delta(T-s)}d\widehat{W_2}(s) \\
&+ \sigma_{\delta 3}\int_t^T e^{-\theta_\delta(T-s)}d\widehat{W_3}(s). \qquad (4.27)
\end{aligned}$$

By considering the above equation, (4.27), the term $\int_t^T e^{-\theta_\delta(T-s)}\kappa_i(s)ds$, where $i = 1, 2$ and 3, can be derived by the same method of integrating factor applied to the equation (4.16), namely

$$\begin{aligned}\int_t^T e^{-\theta_\delta(T-s)}\kappa_i(s)ds &= \frac{\bar{\theta}_{\kappa i}\bar{\mu}_{\kappa i}}{\theta_\delta - \bar{\theta}_{\kappa i}}\int_t^T e^{-\theta_\delta(T-s)}ds - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa i}}\kappa_i(t)\\ &+ \frac{1}{\theta_\delta - \bar{\theta}_{\kappa i}}e^{-\theta_\delta(T-t)}\kappa_i(t)\frac{\sigma_{\kappa i}}{\theta_\delta - \bar{\theta}_{\kappa i}}\int_t^T e^{-\theta_\delta(T-s)}d\widehat{W}_i(s). \quad (4.28)\end{aligned}$$

Since we already solved κ_i, substituting (4.24) into (4.28) yields

$$\begin{aligned}\int_t^T e^{-\theta_\delta(T-s)}\kappa_i(s)ds &= \frac{1}{\theta_\delta - \bar{\theta}_{\kappa i}}\Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa i}(T-t)}\Big)\kappa_i(t)\\ &\frac{\bar{\mu}_{\kappa i}\bar{\theta}_{\kappa i}}{\theta_\delta(\theta_\delta - \bar{\theta}_{\kappa i})}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \frac{\bar{\mu}_{\kappa i}}{\theta_\delta - \bar{\theta}_{\kappa i}}\Big(1 - e^{-\bar{\theta}_{\kappa i}(T-t)}\Big)\\ &\frac{\sigma_{\kappa i}}{\theta_\delta - \bar{\theta}_{\kappa i}}\int_t^T \Big(e^{-\theta_\delta(T-s)} - e^{-\bar{\theta}_{\kappa i}(T-s)}\Big)d\widehat{W}_i(s). \quad (4.29)\end{aligned}$$

Substituting $\int_t^T e^{-\theta_\delta(T-s)}\kappa_i(s)ds$ back into (4.27), we have

$$\begin{aligned}\delta(T) &= \delta(t)e^{-\theta_\delta(T-t)} + \mu_\delta\Big(1 - e^{-\theta_\delta(T-t)}\Big)\\ &+ \frac{\theta_\xi\mu_\xi}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \theta_\xi\int_t^T e^{-\theta_\delta(T-s)}\xi(s)ds\\ &- \sigma_{\delta 1}\frac{\bar{\mu}_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\ &- \sigma_{\delta 2}\frac{\bar{\mu}_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\ &- \sigma_{\delta 3}\frac{\bar{\mu}_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right)\\ &- \sigma_{\delta 1}\frac{1}{\theta_\delta - \bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\kappa_1(t)\\ &- \sigma_{\delta 2}\frac{1}{\theta_\delta - \bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\kappa_2(t)\\ &- \sigma_{\delta 3}\frac{1}{\theta_\delta - \bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\kappa_3(t)\end{aligned}$$

$$
\begin{aligned}
&+ \sigma_{\delta 1} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)} \right) d\widehat{W}_1(s) \right. \\
&+ \sigma_{\delta 2} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)} \right) d\widehat{W}_2(s) \right. \\
&+ \sigma_{\delta 3} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)} \right) d\widehat{W}_3(s) \right.
\end{aligned}
\tag{4.30}
$$

Since there is an integral term, $\int_t^T e^{-\theta_\delta (T-s)} \xi(s) ds$, in the equation (4.30), we need to simplify this term by the method as follows.

By using the integration by parts technique, we have

$$
\int_t^T e^{-\theta_\delta (T-s)} \xi(s) ds = \left[\frac{1}{\theta_\delta} \xi(s) e^{-\theta_\delta (T-s)} \right]_t^T - \frac{1}{\theta_\delta} \int_t^T e^{-\theta_\delta (T-s)} d\xi(s). \tag{4.31}
$$

Now we consider $\int_t^T e^{-\theta_\delta (T-s)} d\xi(s)$ below.

$$
\begin{aligned}
\int_t^T e^{-\theta_\delta (T-s)} d\xi(s) = &\int_t^T e^{-\theta_\delta (T-s)} \theta_\xi \left(\mu_\xi - \frac{1}{\theta_\xi} \left(\sigma_{\xi 1} \kappa_1(s) + \sigma_{\xi 2} \kappa_2 + \sigma_{\xi 3} \kappa_3 \right) - \xi(s) \right) ds \\
&+ \int_t^T e^{-\theta_\delta (T-s)} \sigma_{\xi 1} d\widehat{W}_1(s) \\
&+ \int_t^T e^{-\theta_\delta (T-s)} \sigma_{\xi 2} d\widehat{W}_2(s) \\
&+ \int_t^T e^{-\theta_\delta (T-s)} \sigma_{\xi 3} d\widehat{W}_3(s).
\end{aligned}
\tag{4.32}
$$

Rearranging the above equation, we have

$$
\begin{aligned}
\int_t^T e^{-\theta_\delta(T-s)} d\xi(s) = \theta_\xi \mu_\xi \int_t^T e^{-\theta_\delta(T-s)} ds - \theta_\xi \int_t^T e^{-\theta_\delta(T-s)} \xi(s) ds \\
- \sigma_{\xi 1} \int_t^T e^{-\theta_\delta(T-s)} \kappa_1(s) ds \\
- \sigma_{\xi 2} \int_t^T e^{-\theta_\delta(T-s)} \kappa_2(s) ds \\
- \sigma_{\xi 3} \int_t^T e^{-\theta_\delta(T-s)} \kappa_3(s) ds \\
+ \sigma_{\xi 1} \int_t^T e^{-\theta_\delta(T-s)} d\widehat{W}_1(s) \\
+ \sigma_{\xi 2} \int_t^T e^{-\theta_\delta(T-s)} d\widehat{W}_2(s) \\
+ \sigma_{\xi 3} \int_t^T e^{-\theta_\delta(T-s)} d\widehat{W}_3(s).
\end{aligned}
\tag{4.33}
$$

Substituting (4.29) into (4.33) yields

$$
\begin{aligned}
\int_t^T e^{-\theta_\delta(T-s)} d\xi(s) = \frac{\theta_\xi \mu_\xi}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \theta_\xi \int_t^T e^{-\theta_\delta(T-s)} \xi(s) ds \\
- \sigma_{\xi 1} \frac{\bar{\mu}_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta} \Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big) \right) \\
- \sigma_{\xi 2} \frac{\bar{\mu}_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta} \Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) \right) \\
- \sigma_{\xi 3} \frac{\bar{\mu}_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta} \Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big) \right) \\
- \sigma_{\xi 1} \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 1}} \Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)} \Big) \kappa_1(t) \\
- \sigma_{\xi 2} \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 2}} \Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)} \Big) \kappa_2(t) \\
- \sigma_{\xi 3} \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 3}} \Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)} \Big) \kappa_3(t) \\
+ \sigma_{\xi 1} \int_t^T \left(e^{-\theta_\delta(T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \Big(e^{-\theta_\delta(T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)} \Big) \right) d\widehat{W}_1(s) \\
+ \sigma_{\xi 2} \int_t^T \left(e^{-\theta_\delta(T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \Big(e^{-\theta_\delta(T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)} \Big) \right) d\widehat{W}_2(s) \\
+ \sigma_{\xi 3} \int_t^T \left(e^{-\theta_\delta(T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \Big(e^{-\theta_\delta(T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)} \Big) \right) d\widehat{W}_3(s)
\end{aligned}
\tag{4.34}
$$

From (4.31), we now can evaluate $\left[\frac{1}{\theta_\delta}\xi(s)e^{-\theta_\delta(T-s)}\right]_t^T$ below, namely

$$\left[\frac{1}{\theta_\delta}\xi(s)e^{-\theta_\delta(T-s)}\right]_t^T = \frac{1}{\theta_\delta}\Big(\xi(T)\Big) - \frac{1}{\theta_\delta}\Big(\xi(t)e^{-\theta_\delta(T-t)}\Big). \tag{4.35}$$

Solving the $\xi(T)$ term by using the same approach as to find $\delta(T)$ and then substituting $\xi(T)$ to (4.35), we have

$$\begin{aligned}
\left[\frac{1}{\theta_\delta}\xi(s)e^{-\theta_\delta(T-s)}\right]_t^T &= -\frac{1}{\theta_\delta}\Big(\xi(t)e^{-\theta_\delta(T-t)}\Big)\\
&+ \frac{1}{\theta_\delta}\Bigg[\xi(t)e^{-\theta_\xi(T-t)} + \mu_\xi\Big(1-e^{-\theta_\xi(T-t)}\Big)\\
&- \sigma_{\xi 1}\frac{\bar{\mu}_{\kappa 1}}{\theta_\xi-\bar{\theta}_{\kappa 1}}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big) - \Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&- \sigma_{\xi 2}\frac{\bar{\mu}_{\kappa 2}}{\theta_\xi-\bar{\theta}_{\kappa 2}}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big) - \Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&- \sigma_{\xi 1}\frac{\bar{\mu}_{\kappa 3}}{\theta_\xi-\bar{\theta}_{\kappa 3}}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big) - \Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right)\\
&- \sigma_{\xi 1}\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\kappa_1(t)\\
&- \sigma_{\xi 2}\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\kappa_2(t)\\
&- \sigma_{\xi 3}\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\kappa_3(t)\\
&+ \sigma_{\xi 1}\int_t^T\left(e^{-\theta_\xi(T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)}\Big)\right)d\widehat{W}_1(s)\\
&+ \sigma_{\xi 2}\int_t^T\left(e^{-\theta_\xi(T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)}\Big)\right)d\widehat{W}_2(s)\\
&+ \sigma_{\xi 3}\int_t^T\left(e^{-\theta_\xi(T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)}\Big)\right)d\widehat{W}_3(s)\Bigg].
\end{aligned} \tag{4.36}$$

Now substituting (4.34) and (4.36) into (4.31) and rearranging the equation, we obtain

$$
\begin{aligned}
(1-\frac{\theta_\xi}{\theta_\delta})\int_t^T e^{-\theta_\delta(T-s)}\xi(s)ds &= \frac{\xi(t)}{\theta_\delta}\Big(e^{-\theta_\xi(T-t)}-e^{-\theta_\delta(T-t)}\Big)+\frac{\mu_\xi}{\theta_\delta}\Big(1-e^{-\theta_\xi(T-t)}\Big)\\
&-\frac{\sigma_{\xi 1}\bar{\mu}_{\kappa 1}}{\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&-\frac{\sigma_{\xi 2}\bar{\mu}_{\kappa 2}}{\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&-\frac{\sigma_{\xi 3}\bar{\mu}_{\kappa 3}}{\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 3})}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right)\\
&-\frac{\sigma_{\xi 1}}{\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 1})}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\kappa_1(t)\\
&-\frac{\sigma_{\xi 2}}{\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 2})}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\kappa_2(t)\\
&-\frac{\sigma_{\xi 3}}{\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 3})}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\kappa_3(t)\\
&+\frac{\sigma_{\xi 1}}{\theta_\delta}\int_t^T\left(e^{-\theta_\xi(T-s)}-\frac{\sigma_{\kappa 1}}{\theta_\delta-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-s)}-e^{-\bar{\theta}_{\kappa 1}(T-s)}\Big)\right)d\widehat{W}_1(s)\\
&+\frac{\sigma_{\xi 2}}{\theta_\delta}\int_t^T\left(e^{-\theta_\xi(T-s)}-\frac{\sigma_{\kappa 2}}{\theta_\delta-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-s)}-e^{-\bar{\theta}_{\kappa 2}(T-s)}\Big)\right)d\widehat{W}_2(s)\\
&+\frac{\sigma_{\xi 3}}{\theta_\delta}\int_t^T\left(e^{-\theta_\xi(T-s)}-\frac{\sigma_{\kappa 3}}{\theta_\delta-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-s)}-e^{-\bar{\theta}_{\kappa 3}(T-s)}\Big)\right)d\widehat{W}_3(s)\\
&-\frac{\theta_\xi\mu_\xi}{\theta_\delta^2}\Big(1-e^{-\theta_\delta(T-t)}\Big)\\
&+\sigma_{\xi 1}\frac{\bar{\mu}_{\kappa 1}}{\theta_\delta^2-\bar{\theta}_{\kappa 1}\theta_\delta}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&+\sigma_{\xi 2}\frac{\bar{\mu}_{\kappa 2}}{\theta_\delta^2-\bar{\theta}_{\kappa 2}\theta_\delta}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&+\sigma_{\xi 3}\frac{\bar{\mu}_{\kappa 3}}{\theta_\delta^2-\bar{\theta}_{\kappa 3}\theta_\delta}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right)\\
&+\sigma_{\xi 1}\frac{1}{\theta_\delta^2-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\kappa_1(t)\\
&+\sigma_{\xi 2}\frac{1}{\theta_\delta^2-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\kappa_2(t)\\
&+\sigma_{\xi 3}\frac{1}{\theta_\delta^2-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\kappa_3(t)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\sigma_{\xi 1}}{\theta_\delta} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \Big(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 1} (T-s)} \Big) \right) d\widehat{W}_1(s) \\
&- \frac{\sigma_{\xi 2}}{\theta_\delta} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \Big(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 2} (T-s)} \Big) \right) d\widehat{W}_2(s) \\
&- \frac{\sigma_{\xi 3}}{\theta_\delta} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \Big(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 3} (T-s)} \Big) \right) d\widehat{W}_3(s). \qquad (4.37)
\end{aligned}
$$

Now we simplify (4.38) to obtain $\int_t^T e^{-\theta_\delta (T-s)} \xi(s) ds$ as follows.

$$
\begin{aligned}
\int_t^T e^{-\theta_\delta (T-s)} \xi(s) ds &= \frac{\xi(t)}{\theta_\delta - \theta_\xi} \Big(e^{-\theta_\xi (T-t)} - e^{-\theta_\delta (T-t)} \Big) \\
&+ \frac{\mu_\xi}{\theta_\delta - \theta_\xi} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \frac{\theta_\xi \mu_\xi}{\theta_\delta} \Big(1 - e^{-\theta_\delta (T-t)} \Big) \\
&- \frac{\sigma_{\xi 1} \bar{\mu}_{\kappa 1}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 1})} \left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\xi} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1} (T-t)} \Big) \right) \\
&+ \frac{\sigma_{\xi 1} \bar{\mu}_{\kappa 1}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 1})} \left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1} (T-t)} \Big) \right) \\
&- \frac{\sigma_{\xi 2} \bar{\mu}_{\kappa 2}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 2})} \left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\xi} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2} (T-t)} \Big) \right) \\
&+ \frac{\sigma_{\xi 2} \bar{\mu}_{\kappa 2}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 2})} \left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2} (T-t)} \Big) \right) \\
&- \frac{\sigma_{\xi 3} \bar{\mu}_{\kappa 3}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 3})} \left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 3} (T-t)} \Big) \right) \\
&+ \frac{\sigma_{\xi 3} \bar{\mu}_{\kappa 3}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 3})} \left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta} \Big(1 - e^{-\theta_\xi (T-t)} \Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 3} (T-t)} \Big) \right) \\
&- \frac{\sigma_{\xi 1}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 1})} \Big(e^{-\theta_\xi (T-t)} - e^{-\bar{\theta}_{\kappa 1} (T-t)} \Big) \kappa_1(t) \\
&+ \frac{\sigma_{\xi 1}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 1})} \Big(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 1} (T-t)} \Big) \kappa_1(t) \\
&- \frac{\sigma_{\xi 2}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 2})} \Big(e^{-\theta_\xi (T-t)} - e^{-\bar{\theta}_{\kappa 2} (T-t)} \Big) \kappa_2(t) \\
&+ \frac{\sigma_{\xi 2}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 2})} \Big(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 2} (T-t)} \Big) \kappa_2(t) \\
&- \frac{\sigma_{\xi 3}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 3})} \Big(e^{-\theta_\xi (T-t)} - e^{-\bar{\theta}_{\kappa 3} (T-t)} \Big) \kappa_3(t) \\
&+ \frac{\sigma_{\xi 3}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 3})} \Big(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 3} (T-t)} \Big) \kappa_3(t)
\end{aligned}
$$

$$+\frac{\sigma_{\xi 1}}{(\theta_\delta-\theta_\xi)}\int_t^T\left(e^{-\theta_\xi(T-s)}-\frac{\sigma_{\kappa 1}}{\theta_\delta-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-s)}-e^{-\bar{\theta}_{\kappa 1}(T-s)}\Big)\right)d\widehat{W}_1(s)$$
$$-\frac{\sigma_{\xi 1}}{(\theta_\delta-\theta_\xi)}\int_t^T\left(e^{-\theta_\delta(T-s)}-\frac{\sigma_{\kappa 1}}{\theta_\delta-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\delta(T-s)}-e^{-\bar{\theta}_{\kappa 1}(T-s)}\Big)\right)d\widehat{W}_1(s)$$
$$+\frac{\sigma_{\xi 2}}{(\theta_\delta-\theta_\xi)}\int_t^T\left(e^{-\theta_\xi(T-s)}-\frac{\sigma_{\kappa 2}}{\theta_\delta-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-s)}-e^{-\bar{\theta}_{\kappa 2}(T-s)}\Big)\right)d\widehat{W}_2(s)$$
$$-\frac{\sigma_{\xi 2}}{(\theta_\delta-\theta_\xi)}\int_t^T\left(e^{-\theta_\delta(T-s)}-\frac{\sigma_{\kappa 2}}{\theta_\delta-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\delta(T-s)}-e^{-\bar{\theta}_{\kappa 2}(T-s)}\Big)\right)d\widehat{W}_2(s)$$
$$+\frac{\sigma_{\xi 3}}{(\theta_\delta-\theta_\xi)}\int_t^T\left(e^{-\theta_\xi(T-s)}-\frac{\sigma_{\kappa 3}}{\theta_\delta-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-s)}-e^{-\bar{\theta}_{\kappa 3}(T-s)}\Big)\right)d\widehat{W}_3(s)$$
$$-\frac{\sigma_{\xi 3}}{(\theta_\delta-\theta_\xi)}\int_t^T\left(e^{-\theta_\delta(T-s)}-\frac{\sigma_{\kappa 3}}{\theta_\delta-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\delta(T-s)}-e^{-\bar{\theta}_{\kappa 3}(T-s)}\Big)\right)d\widehat{W}_3(s). \quad (4.38)$$

Now we substitute (4.38) into (4.30) for finding $\delta(T)$, namely

$$\begin{aligned}
\delta(T) &= \delta(t)e^{-\theta_\delta(T-t)}+\mu_\delta\Big(1-e^{-\theta_\delta(T-t)}\Big)+\frac{\theta_\xi\mu_\xi}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)\\
&-\frac{\theta_\xi\xi(t)}{\theta_\delta-\theta_\xi}\Big(e^{-\theta_\xi(T-t)}-e^{-\theta_\delta(T-t)}\Big)\\
&-\frac{\theta_\xi\mu_\xi}{\theta_\delta-\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)+\frac{\theta_\xi^2\mu_\xi}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)\\
&-\frac{\sigma_{\delta 1}\bar{\mu}_{\kappa 1}}{\theta_\delta-\bar{\theta}_{\kappa 1}}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&+\frac{\theta_\xi\sigma_{\xi 1}\bar{\mu}_{\kappa 1}}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&-\frac{\theta_\xi\sigma_{\xi 1}\bar{\mu}_{\kappa 1}}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&-\frac{\sigma_{\delta 2}\bar{\mu}_{\kappa 2}}{\theta_\delta-\bar{\theta}_{\kappa 2}}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&+\frac{\theta_\xi\sigma_{\xi 2}\bar{\mu}_{\kappa 2}}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&-\frac{\theta_\xi\sigma_{\xi 2}\bar{\mu}_{\kappa 2}}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&-\frac{\sigma_{\delta 3}\bar{\mu}_{\kappa 3}}{\theta_\delta-\bar{\theta}_{\kappa 3}}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\Big(1-e^{-\theta_\delta(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\theta_\xi \sigma_{\xi 3} \bar{\mu}_{\kappa 3}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 3})} \left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi} \left(1 - e^{-\theta_\xi (T-t)} \right) - \left(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)} \right) \right) \\
&- \frac{\theta_\xi \sigma_{\xi 3} \bar{\mu}_{\kappa 3}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 3})} \left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta} \left(1 - e^{-\theta_\xi (T-t)} \right) - \left(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)} \right) \right) \\
&- \frac{\sigma_{\delta 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \left(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)} \right) \kappa_1(t) \\
&+ \frac{\theta_\xi \sigma_{\xi 1}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 1})} \left(e^{-\theta_\xi (T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)} \right) \kappa_1(t) \\
&+ \frac{\theta_\xi \sigma_{\xi 1}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 1})} \left(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)} \right) \kappa_1(t) \\
&- \frac{\sigma_{\delta 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \left(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)} \right) \kappa_2(t) \\
&+ \frac{\theta_\xi \sigma_{\xi 2}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 2})} \left(e^{-\theta_\xi (T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)} \right) \kappa_2(t) \\
&+ \frac{\theta_\xi \sigma_{\xi 2}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 2})} \left(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)} \right) \kappa_2(t) \\
&- \frac{\sigma_{\delta 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \left(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)} \right) \kappa_3(t) \\
&+ \frac{\theta_\xi \sigma_{\xi 3}}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 3})} \left(e^{-\theta_\xi (T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)} \right) \kappa_3(t) \\
&+ \frac{\theta_\xi \sigma_{\xi 3}}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 3})} \left(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)} \right) \kappa_3(t) \\
&+ \sigma_{\delta 1} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)} \right) \right) d\widehat{W}_1(s) \\
&- \frac{\theta_\xi \sigma_{\xi 1}}{\theta_\delta - \theta_\xi} \int_t^T \left(e^{-\theta_\xi (T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \left(e^{-\theta_\xi (T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)} \right) \right) d\widehat{W}_1(s) \\
&+ \frac{\theta_\xi \sigma_{\xi 1}}{\theta_\delta - \theta_\xi} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 1}}{\theta_\delta - \bar{\theta}_{\kappa 1}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)} \right) \right) d\widehat{W}_1(s) \\
&+ \sigma_{\delta 2} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)} \right) \right) d\widehat{W}_2(s) \\
&- \frac{\theta_\xi \sigma_{\xi 2}}{\theta_\delta - \theta_\xi} \int_t^T \left(e^{-\theta_\xi (T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \left(e^{-\theta_\xi (T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)} \right) \right) d\widehat{W}_2(s) \\
&+ \frac{\theta_\xi \sigma_{\xi 2}}{\theta_\delta - \theta_\xi} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 2}}{\theta_\delta - \bar{\theta}_{\kappa 2}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)} \right) \right) d\widehat{W}_2(s) \\
&+ \sigma_{\delta 3} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)} \right) \right) d\widehat{W}_3(s) \\
&- \frac{\theta_\xi \sigma_{\xi 3}}{\theta_\delta - \theta_\xi} \int_t^T \left(e^{-\theta_\xi (T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \left(e^{-\theta_\xi (T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)} \right) \right) d\widehat{W}_3(s) \\
&+ \frac{\theta_\xi \sigma_{\xi 3}}{\theta_\delta - \theta_\xi} \int_t^T \left(e^{-\theta_\delta (T-s)} - \frac{\sigma_{\kappa 3}}{\theta_\delta - \bar{\theta}_{\kappa 3}} \left(e^{-\theta_\delta (T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)} \right) \right) d\widehat{W}_3(s). \qquad (4.39)
\end{aligned}
$$

The total earning yield, $\int_t^T \xi(s)ds$, is the only term left that we need to solve for (4.26) by using the same technique as solving the term $\int_t^T \delta(s)ds$ and it can be described mathematically by the following solution

$$
\begin{aligned}
\int_t^T \xi(s)ds &= \frac{1}{\theta_\xi}\xi(t)\Big(1 - e^{-\theta_\xi(T-t)}\Big) + \mu_\xi\bigg((T-t) - \frac{1}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\bigg) \\
&- \frac{\sigma_{\xi 1}\bar{\mu}_{\kappa 1}}{\theta_\xi}\bigg[(T-t) - \frac{1}{\bar{\theta}_{\kappa 1}}\Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big) \\
&- \frac{\bar{\theta}_{\kappa 1}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 1})}\Big(1 - e^{-\theta_\xi(T-t)}\Big) + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 1}}\Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\bigg] \\
&- \frac{\sigma_{\xi 2}\bar{\mu}_{\kappa 2}}{\theta_\xi}\bigg[(T-t) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) \\
&- \frac{\bar{\theta}_{\kappa 2}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 2})}\Big(1 - e^{-\theta_\xi(T-t)}\Big) + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\bigg] \\
&- \frac{\sigma_{\xi 3}\bar{\mu}_{\kappa 3}}{\theta_\xi}\bigg[(T-t) - \frac{1}{\bar{\theta}_{\kappa 3}}\Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big) \\
&- \frac{\bar{\theta}_{\kappa 3}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 3})}\Big(1 - e^{-\theta_\xi(T-t)}\Big) + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 3}}\Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\bigg] \\
&+ \frac{\sigma_{\xi 1}}{\theta_\xi}\bigg[\frac{1}{\theta_\xi - \bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 1}}\Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\bigg]\kappa_1(t) \\
&+ \frac{\sigma_{\xi 2}}{\theta_\xi}\bigg[\frac{1}{\theta_\xi - \bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\bigg]\kappa_2(t) \\
&+ \frac{\sigma_{\xi 3}}{\theta_\xi}\bigg[\frac{1}{\theta_\xi - \bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 3}}\Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\bigg]\kappa_3(t) \\
&+ \frac{\sigma_{\xi 1}}{\theta_\xi}\int_t^T\bigg[1 - \frac{\sigma_{\kappa 1}}{\bar{\theta}_{\kappa 1}} + \frac{\sigma_{\kappa 1}}{\bar{\theta}_{\kappa 1}}e^{-\bar{\theta}_{\kappa 1}(T-s)} - e^{-\theta_\xi(T-s)} \\
&+ \frac{\sigma_{\kappa 1}}{\theta_\xi - \bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-s)} - e^{-\bar{\theta}_{\kappa 1}(T-s)}\Big)\bigg]d\widehat{W}_1(s) \\
&+ \frac{\sigma_{\xi 2}}{\theta_\xi}\int_t^T\bigg[1 - \frac{\sigma_{\kappa 2}}{\bar{\theta}_{\kappa 2}} + \frac{\sigma_{\kappa 2}}{\bar{\theta}_{\kappa 2}}e^{-\bar{\theta}_{\kappa 2}(T-s)} - e^{-\theta_\xi(T-s)} \\
&+ \frac{\sigma_{\kappa 2}}{\theta_\xi - \bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-s)} - e^{-\bar{\theta}_{\kappa 2}(T-s)}\Big)\bigg]d\widehat{W}_2(s) \\
&+ \frac{\sigma_{\xi 3}}{\theta_\xi}\int_t^T\bigg[1 - \frac{\sigma_{\kappa 3}}{\bar{\theta}_{\kappa 3}} + \frac{\sigma_{\kappa 3}}{\bar{\theta}_{\kappa 3}}e^{-\bar{\theta}_{\kappa 3}(T-s)} - e^{-\theta_\xi(T-s)} \\
&+ \frac{\sigma_{\kappa 3}}{\theta_\xi - \bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-s)} - e^{-\bar{\theta}_{\kappa 3}(T-s)}\Big)\bigg]d\widehat{W}_3(s). \qquad (4.40)
\end{aligned}
$$

The term $\int_t^T \delta(s)ds$ finally can be obtained by substituting (4.39) and (4.40) into (4.26). After rearranging the solution, we obtain

$$
\begin{aligned}
\int_t^T \delta(s)ds =& \frac{\theta_\xi \mu_\xi}{\theta_\delta}(T-t) + \frac{1}{\theta_\delta}\Big(1 - e^{-\theta_\delta (T-t)}\Big)\delta(t) \\
&+ \mu_\delta \left[(T-t) - \frac{1}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big)\right] \\
&+ \frac{\theta_\xi}{\theta_\delta}\left[\frac{1}{\theta_\delta - \theta_\xi}\Big(e^{-\theta_\xi(T-t)} - e^{-\theta_\delta(T-t)}\Big) - \frac{1}{\theta_\xi}(1 - e^{-\theta_\xi(T-t)}\Big)\right]\xi(t) \\
&+ \frac{\theta_\xi \mu_\xi}{\theta_\delta}\left[\frac{1}{\theta_\delta - \theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \frac{1}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big)\right. \\
&\left. - \frac{\theta_\xi}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - (T-t) + \frac{1}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\right] \\
&+ \frac{\sigma_{\delta 1}\bar{\mu}_{\kappa 1}}{\theta_\delta}\left[(T-t) - \frac{1}{\bar{\theta}_{\kappa 1}}\big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\big)\right. \\
&\left. - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 1}}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\right] \\
&+ \frac{\sigma_{\delta 2}\bar{\mu}_{\kappa 2}}{\theta_\delta}\left[(T-t) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right. \\
&\left. - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 2}}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\right] \\
&+ \frac{\sigma_{\delta 3}\bar{\mu}_{\kappa 3}}{\theta_\delta}\left[(T-t) - \frac{1}{\bar{\theta}_{\kappa 3}}\Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\right. \\
&\left. - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 3}}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - \Big(1 - e^{-\theta_{\kappa 3}(T-t)}\Big)\right)\right] \\
&+ \frac{\sigma_{\xi 1}\theta_\xi \bar{\mu}_{\kappa 1}}{\theta_\delta}\left[\frac{1}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\right. \\
&- \frac{1}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right) \\
&- \frac{1}{\theta_\xi}\left((T-t) - \frac{1}{\bar{\theta}_{\kappa 1}}\Big(1 - e^{-\theta_{\kappa 1}(T-t)}\Big) - \frac{\bar{\theta}_{\kappa 1}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 1})}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\right. \\
&\left.\left. + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 1}}\Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\right] \\
&+ \frac{\sigma_{\xi 2}\theta_\xi \bar{\mu}_{\kappa 2}}{\theta_\delta}\left[\frac{1}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\right. \\
&- \frac{1}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right) \\
&- \frac{1}{\theta_\xi}\left((T-t) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\theta_{\kappa 2}(T-t)}\Big) - \frac{\bar{\theta}_{\kappa 2}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 2})}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\right. \\
&\left.\left. + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\sigma_{\xi 3}\theta_\xi\bar{\mu}_{\kappa 3}}{\theta_\delta}\Bigg[\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 3})}\Bigg(\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\Bigg)\\
&-\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}\Bigg(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\Big(1-e^{-\theta_\xi(T-t)}\Big)-\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\Bigg)\\
&-\frac{1}{\theta_\xi}\Bigg((T-t)-\frac{1}{\bar{\theta}_{\kappa 3}}\Big(1-e^{-\theta_{\kappa 3}(T-t)}\Big)-\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi(\theta_\xi-\bar{\theta}_{\kappa 3})}\Big(1-e^{-\theta_\xi(T-t)}\Big)\\
&+\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 3}}\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\Bigg)\Bigg]\\
&+\frac{\sigma_{\delta 1}}{\theta_\delta}\Bigg[\frac{1}{\theta_\delta-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)-\frac{1}{\bar{\theta}_{\kappa 1}}\Big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\Bigg]\kappa_1(t)\\
&+\frac{\sigma_{\delta 2}}{\theta_\delta}\Bigg[\frac{1}{\theta_\delta-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)-\frac{1}{\bar{\theta}_{\kappa 2}}\Big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\Bigg]\kappa_2(t)\\
&+\frac{\sigma_{\delta 3}}{\theta_\delta}\Bigg[\frac{1}{\theta_\delta-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)-\frac{1}{\bar{\theta}_{\kappa 3}}\Big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\Bigg]\kappa_3(t)\\
&+\frac{\sigma_{\xi 1}\theta_\xi}{\theta_\delta}\Bigg[\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 1})}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\\
&-\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 1})}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\\
&-\frac{1}{\theta_\xi}\Bigg(\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)-\frac{1}{\bar{\theta}_{\kappa 1}}\big(1-e^{-\bar{\theta}_{\kappa 1}(T-t)}\big)\Bigg)\Bigg]\kappa_1(t)\\
&+\frac{\sigma_{\xi 2}\theta_\xi}{\theta_\delta}\Bigg[\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 2})}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\\
&-\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 2})}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\\
&-\frac{1}{\theta_\xi}\Bigg(\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)-\frac{1}{\bar{\theta}_{\kappa 2}}\big(1-e^{-\bar{\theta}_{\kappa 2}(T-t)}\big)\Bigg)\Bigg]\kappa_2(t)\\
&+\frac{\sigma_{\xi 3}\theta_\xi}{\theta_\delta}\Bigg[\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}\Big(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\\
&-\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 3})}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\\
&-\frac{1}{\theta_\xi}\Bigg(\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)-\frac{1}{\bar{\theta}_{\kappa 3}}\big(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\big)\Bigg)\Bigg]\kappa_3(t)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^T\Bigg[\Big(\frac{\sigma_{\delta 1}}{\theta_\delta}-\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta\bar{\theta}_{\kappa 1}}-\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\xi\theta_\delta}+\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 1}}\Big)\\
&+\Big(\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta\bar{\theta}_{\kappa 1}}-\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 1})}-\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 1}}+\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 1})}\Big)e^{-\bar{\theta}_{\kappa 1}(T-s)}\\
&+\Big(\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 1})}-\frac{\sigma_{\delta 1}}{\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\delta(\theta_\delta-\theta_\xi)}+\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 1})}\Big)e^{-\theta_\delta(T-s)}\\
&+\Big(\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\delta(\theta_\delta-\theta_\xi)}-\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 1})}+\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\xi\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 1})}\Big)e^{-\theta_\xi(T-s)}\Bigg]d\widehat{W}_1(s)\\
&+\int_t^T\Bigg[\Big(\frac{\sigma_{\delta 2}}{\theta_\delta}-\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta\bar{\theta}_{\kappa 2}}-\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\xi\theta_\delta}+\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 2}}\Big)\\
&+\Big(\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta\bar{\theta}_{\kappa 2}}-\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 2})}-\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 2}}+\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 2})}\Big)e^{-\bar{\theta}_{\kappa 2}(T-s)}\\
&+\Big(\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 2})}-\frac{\sigma_{\delta 2}}{\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\delta(\theta_\delta-\theta_\xi)}+\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 2})}\Big)e^{-\theta_\delta(T-s)}\\
&+\Big(\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\delta(\theta_\delta-\theta_\xi)}-\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 2})}+\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\xi\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 2})}\Big)e^{-\theta_\xi(T-s)}\Bigg]d\widehat{W}_2(s)\\
&+\int_t^T\Bigg[\Big(\frac{\sigma_{\delta 3}}{\theta_\delta}-\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta\bar{\theta}_{\kappa 3}}-\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\xi\theta_\delta}+\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 3}}\Big)\\
&+\Big(\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta\bar{\theta}_{\kappa 3}}-\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 3})}-\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 3}}+\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 3})}\Big)e^{-\bar{\theta}_{\kappa 3}(T-s)}\\
&+\Big(\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 3})}-\frac{\sigma_{\delta 3}}{\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\delta(\theta_\delta-\theta_\xi)}+\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}\Big)e^{-\theta_\delta(T-s)}\\
&+\Big(\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\delta(\theta_\delta-\theta_\xi)}-\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}+\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\xi\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 3})}\Big)e^{-\theta_\xi(T-s)}\Bigg]d\widehat{W}_3(s).
\end{aligned}
\tag{4.41}
$$

By substituting (4.41) into (4.19), $S(T)$ is solved and the formula for the stock pricing can be derived.

From the structure of stock distribution, a stock value at maturity $S(T)$ depends on not only the basic well-known parameters: current stock value $S(t)$, volatility σ_S, dividend yield $\delta(t)$, interest rate r, current time t and maturity time T, but also the earning yield $\xi(t)$ (where the dividend yield $\delta(t)$ depends on the earning yield $\xi(t)$). The stock valuation model is derived by the analytical approach of the new formation of the relationship. The following proposition is proposed.

Proposition 4.1. *The stock price at maturity, $S(T)$, with dividend yield $\delta(t)$ and earning yield $\xi(t)$, can be valuated by the following formula.*

$$
\begin{aligned}
S(T) = S(t)\, exp \Big\{ & (r - \frac{\sigma_S^2}{2})(T-t) - A_0(t)\delta(t) - A_1(t) - B_0(t)\xi(t) - B_1(t) - D(t) \\
& + \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) + \sigma_{\delta 2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big) + \sigma_{\delta 3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) \\
& + \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa 1(t)\Big) + \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa 2(t)\Big) + \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa 3(t)\Big) \\
& + \int_t^T \varepsilon_1(s) d\widehat{W}_1(s) + \int_t^T \varepsilon_2(s) d\widehat{W}_2(s) + \int_t^T \varepsilon_3(s) d\widehat{W}_3(s) \Big\} \qquad (4.42)
\end{aligned}
$$

where $W(t)$ denotes the GOU wiener process and $A_i(t)$, $B_i(t)$, $D(t)$, and $\epsilon_i(s)$, where $i = 0, 1, 2, ..., 7$, are as follows.

$$
\begin{aligned}
A_0(t) &= \frac{1}{\theta_\delta}\Big(1 - e^{-\theta_\delta (T-t)}\Big), \\
A_1(t) &= \mu_\delta \Big[(T-t) - \frac{1}{\theta_\delta}\Big(1 - e^{-\theta_\delta (T-t)}\Big)\Big], \\
A_2(t) &= \frac{\bar{\mu}_{\kappa 1}}{\theta_\delta}\Big[(T-t) - \frac{1}{\bar{\theta}_{\kappa 1}}\big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\big) \\
&\quad - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 1}}\Big(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1 - e^{-\theta_\delta (T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\Big)\Big], \\
A_3(t) &= \frac{1}{\theta_\delta}\Big[\frac{1}{\theta_\delta - \bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 1}}\Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\Big], \\
A_4(t) &= \frac{\bar{\mu}_{\kappa 2}}{\theta_\delta}\Big[(T-t) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) \\
&\quad - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 2}}\Big(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1 - e^{-\theta_\delta (T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\Big)\Big], \\
A_5(t) &= \frac{1}{\theta_\delta}\Big[\frac{1}{\theta_\delta - \bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\Big], \\
A_6(t) &= \frac{\bar{\mu}_{\kappa 3}}{\theta_\delta}\Big[(T-t) - \frac{1}{\bar{\theta}_{\kappa 3}}\Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big) \\
&\quad - \frac{1}{\theta_\delta - \bar{\theta}_{\kappa 3}}\Big(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\Big(1 - e^{-\theta_\delta (T-t)}\Big) - \Big(1 - e^{-\theta_{\kappa 3}(T-t)}\Big)\Big)\Big], \\
A_7(t) &= \frac{1}{\theta_\delta}\Big[\frac{1}{\theta_\delta - \bar{\theta}_{\kappa 3}}\Big(e^{-\theta_\delta (T-t)} - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 3}}\Big(1 - e^{-\bar{\theta}_{\kappa 3}(T-t)}\Big)\Big],
\end{aligned}
$$

$$
\begin{aligned}
B_0(t) &= \frac{\theta_\xi}{\theta_\delta}\left[\frac{1}{\theta_\delta - \theta_\xi}\Big(e^{-\theta_\xi(T-t)} - e^{-\theta_\delta(T-t)}\Big) - \frac{1}{\theta_\xi}\big(1 - e^{-\theta_\xi(T-t)}\Big)\right],\\
B_1(t) &= \frac{\theta_\xi\mu_\xi}{\theta_\delta}\left[\frac{1}{\theta_\delta - \theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \frac{1}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big)\right.\\
&\quad \left. - \frac{\theta_\xi}{\theta_\delta}\Big(1 - e^{-\theta_\delta(T-t)}\Big) - (T-t) + \frac{1}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\right],\\
B_2(t) &= \frac{\theta_\xi\bar{\mu}_{\kappa 1}}{\theta_\delta}\left[\frac{1}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\right.\\
&\quad - \frac{1}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 1})}\left(\frac{\bar{\theta}_{\kappa 1}}{\theta_\delta}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\\
&\quad - \frac{1}{\theta_\xi}\left((T-t) - \frac{1}{\bar{\theta}_{\kappa 1}}\Big(1 - e^{-\theta_{\kappa 1}(T-t)}\Big) - \frac{\bar{\theta}_{\kappa 1}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 1})}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\right.\\
&\quad \left.\left. + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 1}}\Big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right)\right],\\
B_3(t) &= \frac{\theta_\xi}{\theta_\delta}\left[\frac{1}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 1})}\Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\right.\\
&\quad - \frac{1}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 1})}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big)\\
&\quad \left. - \frac{1}{\theta_\xi}\left(\frac{1}{\theta_\xi - \bar{\theta}_{\kappa 1}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 1}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 1}}\big(1 - e^{-\bar{\theta}_{\kappa 1}(T-t)}\big)\right)\right],\\
B_4(t) &= \frac{\theta_\xi\bar{\mu}_{\kappa 2}}{\theta_\delta}\left[\frac{1}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\xi}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\right.\\
&\quad - \frac{1}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 2})}\left(\frac{\bar{\theta}_{\kappa 2}}{\theta_\delta}\Big(1 - e^{-\theta_\xi(T-t)}\Big) - \Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\\
&\quad - \frac{1}{\theta_\xi}\left((T-t) - \frac{1}{\bar{\theta}_{\kappa 2}}\Big(1 - e^{-\theta_{\kappa 2}(T-t)}\Big) - \frac{\bar{\theta}_{\kappa 2}}{\theta_\xi(\theta_\xi - \bar{\theta}_{\kappa 2})}\Big(1 - e^{-\theta_\xi(T-t)}\Big)\right.\\
&\quad \left.\left. + \frac{1}{\theta_\xi - \bar{\theta}_{\kappa 2}}\Big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right)\right],\\
B_5(t) &= \frac{\theta_\xi}{\theta_\delta}\left[\frac{1}{(\theta_\delta - \theta_\xi)(\theta_\delta - \bar{\theta}_{\kappa 2})}\Big(e^{-\theta_\delta(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\right.\\
&\quad - \frac{1}{(\theta_\delta - \theta_\xi)(\theta_\xi - \bar{\theta}_{\kappa 2})}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big)\\
&\quad \left. - \frac{1}{\theta_\xi}\left(\frac{1}{\theta_\xi - \bar{\theta}_{\kappa 2}}\Big(e^{-\theta_\xi(T-t)} - e^{-\bar{\theta}_{\kappa 2}(T-t)}\Big) - \frac{1}{\bar{\theta}_{\kappa 2}}\big(1 - e^{-\bar{\theta}_{\kappa 2}(T-t)}\big)\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
B_6(t) &= \frac{\theta_\xi \bar{\mu}_{\kappa 3}}{\theta_\delta}\Bigg[\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 3})}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi}\left(1-e^{-\theta_\xi(T-t)}\right)-\left(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)\right)\\
&-\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}\left(\frac{\bar{\theta}_{\kappa 3}}{\theta_\delta}\left(1-e^{-\theta_\xi(T-t)}\right)-\left(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)\right)\\
&-\frac{1}{\theta_\xi}\Bigg((T-t)-\frac{1}{\bar{\theta}_{\kappa 3}}\left(1-e^{-\theta_{\kappa 3}(T-t)}\right)-\frac{\bar{\theta}_{\kappa 3}}{\theta_\xi(\theta_\xi-\bar{\theta}_{\kappa 3})}\left(1-e^{-\theta_\xi(T-t)}\right)\\
&+\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 3}}\left(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)\Bigg)\Bigg],\\
B_7(t) &= \frac{\theta_\xi}{\theta_\delta}\Bigg[\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}\left(e^{-\theta_\delta(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)\\
&-\frac{1}{(\theta_\delta-\theta_\xi)(\theta_\xi-\bar{\theta}_{\kappa 3})}\left(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)\\
&-\frac{1}{\theta_\xi}\left(\frac{1}{\theta_\xi-\bar{\theta}_{\kappa 3}}\left(e^{-\theta_\xi(T-t)}-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)-\frac{1}{\bar{\theta}_{\kappa 3}}\left(1-e^{-\bar{\theta}_{\kappa 3}(T-t)}\right)\right)\Bigg],\\
D(t) &= \frac{\theta_\xi \mu_\xi}{\theta_\delta}(T-t),\\
\varepsilon_1(s) &= \sigma_S-\Bigg[\left(\frac{\sigma_{\delta 1}}{\theta_\delta}-\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta\bar{\theta}_{\kappa 1}}-\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\xi\theta_\delta}+\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 1}}\right)\\
&+\left(\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta\bar{\theta}_{\kappa 1}}-\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 1})}-\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 1}}+\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 1})}\right)e^{-\bar{\theta}_{\kappa 1}(T-s)}\\
&+\left(\frac{\sigma_{\delta 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 1})}-\frac{\sigma_{\delta 1}}{\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\delta(\theta_\delta-\theta_\xi)}+\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 1})}\right)e^{-\theta_\delta(T-s)}\\
&+\left(\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\delta(\theta_\delta-\theta_\xi)}-\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 1})}+\frac{\theta_\xi\sigma_{\xi 1}}{\theta_\xi\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 1}\sigma_{\kappa 1}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 1})}\right)e^{-\theta_\xi(T-s)}\Bigg],\\
\varepsilon_2(s) &= -\Bigg[\left(\frac{\sigma_{\delta 2}}{\theta_\delta}-\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta\bar{\theta}_{\kappa 2}}-\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\xi\theta_\delta}+\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 2}}\right)\\
&+\left(\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta\bar{\theta}_{\kappa 2}}-\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 2})}-\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 2}}+\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 2})}\right)e^{-\bar{\theta}_{\kappa 2}(T-s)}\\
&+\left(\frac{\sigma_{\delta 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 2})}-\frac{\sigma_{\delta 2}}{\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\delta(\theta_\delta-\theta_\xi)}+\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 2})}\right)e^{-\theta_\delta(T-s)}\\
&+\left(\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\delta(\theta_\delta-\theta_\xi)}-\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 2})}+\frac{\theta_\xi\sigma_{\xi 2}}{\theta_\xi\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 2}\sigma_{\kappa 2}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 2})}\right)e^{-\theta_\xi(T-s)}\Bigg],\\
\varepsilon_3(s) &= -\Bigg[\left(\frac{\sigma_{\delta 3}}{\theta_\delta}-\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta\bar{\theta}_{\kappa 3}}-\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\xi\theta_\delta}+\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 3}}\right)\\
&+\left(\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta\bar{\theta}_{\kappa 3}}-\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 3})}-\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta\bar{\theta}_{\kappa 3}}+\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 3})}\right)e^{-\bar{\theta}_{\kappa 3}(T-s)}\\
&+\left(\frac{\sigma_{\delta 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\bar{\theta}_{\kappa 3})}-\frac{\sigma_{\delta 3}}{\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\delta(\theta_\delta-\theta_\xi)}+\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}\right)e^{-\theta_\delta(T-s)}\\
&+\left(\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\delta(\theta_\delta-\theta_\xi)}-\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\delta(\theta_\delta-\theta_\xi)(\theta_\delta-\bar{\theta}_{\kappa 3})}+\frac{\theta_\xi\sigma_{\xi 3}}{\theta_\xi\theta_\delta}-\frac{\theta_\xi\sigma_{\xi 3}\sigma_{\kappa 3}}{\theta_\xi\theta_\delta(\theta_\xi-\bar{\theta}_{\kappa 3})}\right)e^{-\theta_\xi(T-s)}\Bigg].
\end{aligned}
$$

The proposition 1 can be applied in order to derive the formulas for European options pricing by using Ito's lemma and explicit computation.

4.5 The Options Pricing Formula

Options pricing is the main topic in this thesis; therefore, we will determine the formulas to evaluate the option price by using the analytical approach and applying Ito's lemmas.

4.5.1 European Call Option

Call option is the main formula to derive since Put option can be derived by using the Put-Call parity relationship. Generally, option pricing is impacted by the stock price. From previous section, we proposed the formula of stock pricing. In order to solve the European options formula, the Ito isometry lemma is applied to solve the European options formula

$$E\Big[\Big(\int_t^T \varepsilon_i(s)d\widehat{W}_i(s)\Big)^2\Big] = E\Big[\int_t^T \varepsilon_i(s)^2 ds\Big]. \tag{4.43}$$

From (4.43), we define the expectation of the movement of our stock pricing model as follows

$$\Sigma(t)^2 = \int_t^T \varepsilon_1(s)^2 ds + \int_t^T \varepsilon_2(s)^2 ds + \int_t^T \varepsilon_3(s)^2 ds. \tag{4.44}$$

Now, from (4.18), we need to solve for $E^Q\Big[S(T)\mathbf{1}_{S(T)>K}|F_t\Big]$ and $E^Q\Big[\mathbf{1}_{S(T)>K}|F_t\Big]$ in order to get the Call option price. Thus, by following Lioui's approach for analytically solving for option price [38, 38], we let

$$\begin{aligned}
\Phi(t) = {} & A_0(t)\delta(t) + A_1(t) + B_0(t)\xi(t) + B_1(t) + D(t) \\
& - \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) - \sigma_{\delta_2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big) \\
& - \sigma_{\delta_3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) - \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa_1(t)\Big) \\
& - \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa_2(t)\Big) - \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa_3(t)\Big).
\end{aligned} \tag{4.45}$$

Note that when call option is exercised, $S(T) > K$, under the risk-adjusted probabilities, we denote η to follow a standard normal distribution and define

$$E\Big[\Sigma(t)^2\Big] = \eta\Sigma(t). \tag{4.46}$$

From proposition 1, we have

$$-\eta \leq \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big). \tag{4.47}$$

Denote

$$d_2 = \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big). \tag{4.48}$$

Thus (4.47) and (4.48) gives $-\eta \leq d_2$. The risk-adjusted probability of this event is $1 - N(-d_2) = N(d_2)$ [76]. This yields $E^Q\Big[\mathbf{1}_{S(T)>K}|F_t\Big]$ with

$$E^Q\Big[\mathbf{1}_{S(T)>K}|F_t\Big] = N(d_2), \tag{4.49}$$

where $d_2 = \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big)$.

Next, we derive for $E^Q\Big[S(T)\mathbf{1}_{S(T)>K}|F_t\Big]$ as follows.

$$\begin{aligned}
E^Q\Big[S(T)\mathbf{1}_{S(T)>K}|F_t\Big] &= E^Q\Big[S(T)\mathbf{1}_{\eta>-d_2}|F_t\Big] \\
&= S(t)\ exp\Big\{(r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big\}E^Q\Big[e^{\Sigma(t)\eta}\mathbf{1}_{\eta>-d_2}|F_t\Big] \\
&= S(t)e^{(r-\frac{\sigma_s^2}{2})(T-t)-\Phi(t)}e^{\frac{1}{2}\Sigma(t)^2}N\big(d_2 + \Sigma(t)\big) \\
&= S(t)e^{(r-\frac{\sigma_s^2}{2})(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2}N(d_1),
\end{aligned} \tag{4.50}$$

since $E^Q\Big[e^{\Sigma(t)\eta}\mathbf{1}_{\eta>-d_2}|F_t\Big] = e^{\frac{1}{2}\Sigma(t)^2}N\big(d_2 + \Sigma(t)\big)$ and $d_1 = d_2 + \Sigma(t)$.

By applying (4.49) and (4.50) to (4.18), the European call option pricing formula can be derived as detailed below

European call option formula $C(t)$ is derived when stock price $S(t)$ is matured at time T and with the exercise price K. By given the corresponding conditions of dividend yield $\delta(t)$ and earning yield $\xi(t)$, the explicit formula is given in the following proposition.

Proposition 4.2. *The price of European call option $C(t)$, of maturity T written on stock $S(t)$ with exercise price K, stochastic dividend $\delta(t)$ and stochastic earning yield $\xi(t)$, is*

$$C(t) = S(t)e^{\left[-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right]}N(d_1) - Ke^{-r(T-t)}N(d_2) \tag{4.51}$$

where

$$\begin{aligned}
d_1 =& d_2 + \Sigma(t),\\
d_2 =& \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big),\\
\Sigma(t)^2 =& \int_t^T \varepsilon_1(s)^2 ds + \int_t^T \varepsilon_2(s)^2 ds + \int_t^T \varepsilon_3(s)^2 ds,\\
\Phi(t) =& A_0(t)\delta(t) + A_1(t) + B_0(t)\xi(t) + B_1(t) + D(t)\\
&- \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) - \sigma_{\delta_2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big)\\
&- \sigma_{\delta_3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) - \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa_1(t)\Big)\\
&- \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa_2(t)\Big) - \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa_3(t)\Big),
\end{aligned}$$

and $A_0(t) - A_7(t)$, $B_0(t) - B_7(t)$, $D(t)$ and $\varepsilon_1(s) - \varepsilon_3(s)$ are precisely given in the proposition 1.

4.5.2 Put-Call Parity

In financial mathematics, Put-Call parity defines a relationship between the price of a European call option and European put option written on a stock $S(t)$ with the same maturity and exercise price in a frictionless market. For the portfolio of buying a call option $C(t)$ and selling a put option $P(t)$ for the stock price $S(t)$ at maturity T and exercise price K, in the absence of arbitrage opportunities, the following relationship exists between the value of the various instruments

$$\begin{aligned}
C(t) - P(t) &= e^{-r(T-t)}E^Q\Big[S(T) - K|F_t\Big]\\
&= e^{-r(T-t)}E^Q\Big[S(T)|F_t\Big] - Ke^{-r(T-t)}
\end{aligned} \tag{4.52}$$

when the $e^{-r(T-t)}$ is explicitly assigned as a discount factor to time t. By applying equation (4.42) in proposition 1 , $E^Q\Big[S(T)|F_t\Big]$ is derived

$$E^Q\Big[S(T)|F_t\Big] = S(t)\, exp\Big\{(r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big\}E^Q\Big[e^{\Sigma(t)\eta}|F_t\Big] \tag{4.53}$$

$$= S(t)\, exp\Big\{(r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t) + \frac{1}{2}\Sigma(t)^2\Big\}. \tag{4.54}$$

Substitute (4.54) into (4.52), then given the corresponding conditions of dividend yield $\delta(t)$ and earning yield $\xi(t)$, the explicit formula of put-call parity is obtained as follows.

Proposition 4.3. *The put-call parity between the European call option $C(t)$ and put option $P(t)$, related to the stochastic dividend yield $\delta(t)$ and stochastic earning yield $\xi(t)$ with exercise price K at maturity T, is*

$$C(t) - P(t) = S(t)e^{\Big[-\Phi(t) - \frac{\sigma_S^2}{2}(T-t) + \frac{1}{2}\Sigma(t)^2\Big]} - Ke^{-r(T-t)} \tag{4.55}$$

where

$$\begin{aligned}
d_1 =& d_2 + \Sigma(t),\\
d_2 =& \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big),\\
\Sigma(t)^2 =& \int_t^T \varepsilon_1(s)^2 ds + \int_t^T \varepsilon_2(s)^2 ds + \int_t^T \varepsilon_3(s)^2 ds,\\
\Phi(t) =& A_0(t)\delta(t) + A_1(t) + B_0(t)\xi(t) + B_1(t) + D(t)\\
&- \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) - \sigma_{\delta_2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big)\\
&- \sigma_{\delta_3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) - \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa_1(t)\Big)\\
&- \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa_2(t)\Big) - \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa_3(t)\Big),
\end{aligned}$$

and $A_0(t) - A_7(t)$, $B_0(t) - B_7(t)$, $D(t)$ and $\varepsilon_1(s) - \varepsilon_3(s)$ are precisely given in the proposition 1.

4.5.3 European Put Option

As a security for the investors, options of the stock have been determined widely. Not only Call option but also Put option are important as a derivative for investment in the portfolio.

By applying the equation (4.51) in proposition 2 and the equation (4.55) in proposition 3, the European put option formula $P(t)$ is derived when the stock price $S(t)$ is matured at time T with the exercise price K. Given the corresponding conditions of dividend yield $\delta(t)$ and earning yield $\xi(t)$, the explicit formula for the put option is given as follows.

Proposition 4.4. *The price of European put option $C(t)$ of maturity T written on stock $S(t)$ with exercise price K, stochastic dividend $\delta(t)$ and stochastic earning yield $\xi(t)$, is*

$$P(t) = Ke^{-r(T-t)}N(-d_2) - S(t)e^{\left[-\Phi(t)-\frac{\sigma_S^2}{2}(T-t)-\frac{1}{2}\Sigma(t)^2\right]}N(-d_1) \tag{4.56}$$

where

$$\begin{aligned}
d_1 =& d_2 + \Sigma(t), \\
d_2 =& \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big), \\
\Sigma(t)^2 =& \int_t^T \varepsilon_1(s)^2 ds + \int_t^T \varepsilon_2(s)^2 ds + \int_t^T \varepsilon_3(s)^2 ds, \\
\Phi(t) =& A_0(t)\delta(t) + A_1(t) + B_0(t)\xi(t) + B_1(t) + D(t) \\
& - \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) - \sigma_{\delta_2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big) \\
& - \sigma_{\delta_3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) - \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa_1(t)\Big) \\
& - \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa_2(t)\Big) - \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa_3(t)\Big),
\end{aligned}$$

and $A_0(t) - A_7(t)$, $B_0(t) - B_7(t)$, $D(t)$ and $\varepsilon_1(s) - \varepsilon_3(s)$ are precisely given in the proposition 4.1.

4.6 Concluding Remarks

For more than 30 years, the Black-Scholes-Merton model has been playing a very important role in the financial world. The factors which can be taken into account in the model are well-known for decades, except for the P/E ratio or earning yield.

In this chapter, we introduce the new stochastic process, named GOU process, which is the generalization of the Ornstein-Uhlenbeck process. By applying the GOU process, we theoretically extend the pricing model taking into account the P/E ratio or earning yield. The call option price formula (4.51), put-call parity relationship (4.55) and the put option price formula (4.55) are developed with the features of stochastic dividend yield and stochastic earning yield including the stochastic market price of risk(MPR) as an important stochastic factor in the real-world situation. In the next chapter, we will test the sensitivity of our model and examine the model performance to check the possibility to use the new model in the real-world market.

Chapter 5

Greek Parameters of Option Pricing Model with Stochastic Earning Yield

5.1 General

The Greek Parameters are the sensitivity measure to the rate of change of the option price with respect to the parameters of the model. Once the option pricing model has been proposed, the Greek parameters are the quantities which are required to represent the sensitivity of the option price; for instance, change of the option price to a change in the underlying price and change of the option price to a change in the implied volatility. In this chapter, we will propose the proposition for the measure of the risks from changes in some parameters.

The ultimate goal of this thesis is to improve the option pricing model so that it can allow more parameters to be taken into account. In last chapter we theoretically develop a new model and establish the propositions for the European call and put options taking account into the stochastic dividend yield, the stochastic earning yield and the stochastic market price of risk (MPR). The Greek parameters as the sensitivity representative of the option pricing model are the common measures to check the sensitivity of the model to the change of parameters. In this chapter we will theoretically prove and propose the propositions for Greek parameters.

5.2 Delta

Delta is the first-order Greeks parameter which is the measure of the rate of change of the option price with respect to the change in the stock price or underlying asset's price. In other words, Delta can be described as the first order derivative of the option price, $C(t)$, with respect to the underlying asset price, $S(t)$. The delta value, $\Delta_C(t)$, is greater than zero and lower than one [54]. The same meaning is defined for the Delta in the option pricing model with the earning yield parameter by the number of underlying units in a portfolio replicating the option.

In order to derive the formula for the Delta by replicating method with an idea from Chapter 3, we refer to the proposition 4.2. By following the analytical approach of Lioui [38, 39], applying Ito's lemma to equation (4.51), we obtain

$$\begin{aligned} dC(t) =& (\cdot)dt + e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1)dS(t) \\ &- S(t)e^{\left[-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right]} N(d_1)\Big(A_0(t)d\delta(t) + \sigma_{\delta 1}A_3(t)d\kappa_1(t) \\ &+ \sigma_{\delta 2}A_5(t)d\kappa_2(t) + \sigma_{\delta_3}A_7(t)d\kappa_3(t) + B_0(t)d\delta(t) + \sigma_{\xi 1}B_3(t)d\kappa_1(t) \\ &+ \sigma_{\xi 2}B_5(t)d\kappa_2(t) + \sigma_{\xi 3}B_7(t)d\kappa_3(t)\Big). \end{aligned} \tag{5.1}$$

By applying (4.11)-(4.16), (5.1) becomes

$$\begin{aligned} dC(t) =& (\cdot)dt + e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1)dS(t) \\ &\times \left(\begin{bmatrix} I & J & K \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix} \right), \end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
I =& S(t)\sigma_S - A_0(t)S(t)\sigma_{\delta 1} + \sigma_{\delta 1}A_3(t)S(t)\sigma_{\kappa 1} - B_0(t)S(t)\sigma_{\xi 1} + \sigma_{\xi 1}B_3(t)S(t)\sigma_{\kappa 1},\\
J =& - A_0(t)S(t)\sigma_{\delta 2} + \sigma_{\delta 2}A_5(t)S(t)\sigma_{\kappa 2} - B_0(t)S(t)\sigma_{\xi 2} + \sigma_{\xi 2}B_5(t)S(t)\sigma_{\kappa 2},\\
K =& - A_0(t)S(t)\sigma_{\delta 3} + \sigma_{\delta 3}A_5(t)S(t)\sigma_{\kappa 3} - B_0(t)S(t)\sigma_{\xi 3} + \sigma_{\xi 3}B_7(t)S(t)\sigma_{\kappa 3}.
\end{aligned}$$

With the idea given in Chapter 3, we consider the portfolio replication. If there are $\alpha(t)$ units of the riskless asset, we have

$$C(t) = \alpha(t)e^{rt} + \Delta_C(t)S(t) + \beta(t)\big(X(t) + Y(t)\big). \tag{5.3}$$

By the self-financing trading strategy, we get

$$\begin{aligned}
dC(t) =& \alpha(t)d\Big(e^{rt}\Big) + \Delta_C(t)dS(t) + \Delta_C(t)\delta(t)S(t)dt\\
&+ \beta(t)dX(t) + \beta(t)dY(t).
\end{aligned} \tag{5.4}$$

Consequently,

$$dC(t) = (\cdot)dt + \left(\Big[M \cdot N\Big]^{\intercal} \begin{bmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{bmatrix} \right), \tag{5.5}$$

where

$$M = \begin{bmatrix} \sigma_S & \sigma_{X1}(t) & \sigma_{Y1}(t) \\ 0 & \sigma_{X2}(t) & \sigma_{Y2}(t) \\ 0 & \sigma_{X3}(t) & \sigma_{Y3}(t) \end{bmatrix},$$

$$N = \begin{bmatrix} \Delta_C(t)dS(t) \\ \beta(t)X(t) \\ \beta(t)Y(t) \end{bmatrix}.$$

From (5.2) and (5.5), we obtain

$$e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\begin{bmatrix}I & J & K\end{bmatrix}=\begin{bmatrix}M\cdot N\end{bmatrix}^{\intercal}, \tag{5.6}$$

where

$$\begin{aligned}
I =& S(t)\sigma_S - A_0(t)S(t)\sigma_{\delta 1} + \sigma_{\delta 1}A_3(t)S(t)\sigma_{\kappa 1} - B_0(t)S(t)\sigma_{\xi 1} + \sigma_{\xi 1}B_3(t)S(t)\sigma_{\kappa 1},\\
J =& - A_0(t)S(t)\sigma_{\delta 2} + \sigma_{\delta 2}A_5(t)S(t)\sigma_{\kappa 2} - B_0(t)S(t)\sigma_{\xi 2} + \sigma_{\xi 2}B_5(t)S(t)\sigma_{\kappa 2},\\
K =& - A_0(t)S(t)\sigma_{\delta 3} + \sigma_{\delta 3}A_5(t)S(t)\sigma_{\kappa 3} - B_0(t)S(t)\sigma_{\xi 3} + \sigma_{\xi 3}B_7(t)S(t)\sigma_{\kappa 3},
\end{aligned}$$

$$M=\begin{bmatrix}\sigma_S & \sigma_{X1}(t) & \sigma_{Y1}(t)\\ 0 & \sigma_{X2}(t) & \sigma_{Y2}(t)\\ 0 & \sigma_{X3}(t) & \sigma_{Y3}(t)\end{bmatrix},$$

$$N=\begin{bmatrix}\Delta_C(t)dS(t)\\ \beta(t)X(t)\\ \beta(t)Y(t)\end{bmatrix},$$

Then we derive

$$\begin{bmatrix}M\cdot N\end{bmatrix}^{\intercal}=\begin{bmatrix}\sigma_S\Delta_C(t)S(t)+\sigma_{X1}(t)\beta(t)X(t)+\sigma_{Y1}(t)\beta(t)Y(t)\\ \sigma_{X2}(t)\beta(t)X(t)+\sigma_{Y2}(t)\beta(t)Y(t)\\ \sigma_{X3}(t)\beta(t)X(t)+\sigma_{Y3}(t)\beta(t)Y(t)\\ \\ \end{bmatrix}. \tag{5.7}$$

From (5.6) and (5.7), we have

$$e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[I]=\sigma_S\Delta_C(t)S(t)+\sigma_{X1}(t)\beta(t)X(t)+\sigma_{Y1}(t)\beta(t)Y(t), \tag{5.8}$$

$$e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[J]=\sigma_{X2}(t)\beta(t)X(t)+\sigma_{Y2}(t)\beta(t)Y(t), \tag{5.9}$$

$$e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[K]=\sigma_{X3}(t)\beta(t)X(t)+\sigma_{Y3}(t)\beta(t)Y(t), \tag{5.10}$$

where

$$\begin{aligned}
I =& S(t)\sigma_S - A_0(t)S(t)\sigma_{\delta 1} + \sigma_{\delta 1}A_3(t)S(t)\sigma_{\kappa 1} - B_0(t)S(t)\sigma_{\xi 1} + \sigma_{\xi 1}B_3(t)S(t)\sigma_{\kappa 1},\\
J =& - A_0(t)S(t)\sigma_{\delta 2} + \sigma_{\delta 2}A_5(t)S(t)\sigma_{\kappa 2} - B_0(t)S(t)\sigma_{\xi 2} + \sigma_{\xi 2}B_5(t)S(t)\sigma_{\kappa 2},\\
K =& - A_0(t)S(t)\sigma_{\delta 3} + \sigma_{\delta 3}A_5(t)S(t)\sigma_{\kappa 3} - B_0(t)S(t)\sigma_{\xi 3} + \sigma_{\xi 3}B_7(t)S(t)\sigma_{\kappa 3},
\end{aligned}$$

Now, multiplying through equation (5.9) by $\frac{-\sigma_{X3}}{\sigma_{X2}}$ and adding the result into (5.10), we derive

$$\frac{-\sigma_{X3}}{\sigma_{X2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\Big([J]+[K]\Big) = \Big(\frac{-\sigma_{X3}\sigma_{Y2}}{\sigma_{X2}}+\sigma_{Y3}\Big)\beta(t)Y(t) \tag{5.11}$$

By simplification, we get

$$\beta(t)Y(t) = \frac{-\sigma_{X3}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\Big([J]+[K]\Big) \tag{5.12}$$

Substituting $\beta(t)Y(t)$ into (5.9), we obtain

$$\begin{aligned}
\beta(t)X(t) =& \frac{\sigma_{Y3}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[J]\\
&- \frac{\sigma_{Y2}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[K].
\end{aligned} \tag{5.13}$$

Then, we can derive $\sigma_S\Delta_C(t)S(t)$ as follows

$$\begin{aligned}
\sigma_S\Delta_C(t)S(t) =& e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[I]\\
&- \frac{\sigma_{X1}\sigma_{Y3}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[J]\\
&+ \frac{\sigma_{X1}\sigma_{Y2}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[K]\\
&+ \frac{\sigma_{Y1}\sigma_{X3}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[J]\\
&- \frac{\sigma_{Y1}\sigma_{X2}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}N(d_1)\cdot[K]
\end{aligned} \tag{5.14}$$

which, after simplification, becomes

$$\begin{aligned}
\sigma_S \Delta_C(t) S(t) =& e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1)\cdot[I] \\
&+ \frac{\sigma_{Y1}\sigma_{X3}-\sigma_{X1}\sigma_{Y3}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}} e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1)\cdot[J] \\
&+ \frac{\sigma_{X1}\sigma_{Y2}-\sigma_{Y1}\sigma_{X2}}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}} e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1)\cdot[K] \qquad (5.15)
\end{aligned}$$

where

$$\begin{aligned}
I =& S(t)\sigma_S - A_0(t)S(t)\sigma_{\delta 1} + \sigma_{\delta 1}A_3(t)S(t)\sigma_{\kappa 1} - B_0(t)S(t)\sigma_{\xi 1} + \sigma_{\xi 1}B_3(t)S(t)\sigma_{\kappa 1}, \\
J =& -A_0(t)S(t)\sigma_{\delta 2} + \sigma_{\delta 2}A_5(t)S(t)\sigma_{\kappa 2} - B_0(t)S(t)\sigma_{\xi 2} + \sigma_{\xi 2}B_5(t)S(t)\sigma_{\kappa 2}, \\
K =& -A_0(t)S(t)\sigma_{\delta 3} + \sigma_{\delta 3}A_5(t)S(t)\sigma_{\kappa 3} - B_0(t)S(t)\sigma_{\xi 3} + \sigma_{\xi 3}B_7(t)S(t)\sigma_{\kappa 3}.
\end{aligned}$$

Consequently, by simply rearranging the above equation, $\Delta_C(t)$ can be derived as the formula for delta of the call option with stochastic earning yield as the units number of the underlying asset in such a portfolio to replicate the call option. We thus propose the proposition as follows.

Proposition 5.1. *The delta of call option as the units number of the underlying asset in such a portfolio to replicate the call option,*$\Delta_C(t)$ *written on* $S(t)$ *with exercise price* K*, maturity time* T*, stochastic earning yield* $\xi(t)$*, stochastic dividend yield* $\delta(t)$ *and stochastic market price of risk* $\kappa(t)$*, is*

$$\begin{aligned}
\Delta_C(t) =& e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1)\cdot \frac{1}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}} \\
&\times \Big[(\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2})H_1(t) \\
&+ (\sigma_{Y1}\sigma_{X3}-\sigma_{X1}\sigma_{Y3})H_2(t) \\
&+ (\sigma_{X1}\sigma_{Y2}-\sigma_{Y3}\sigma_{X2})H_3(t)\Big], \qquad (5.16)
\end{aligned}$$

where

$$
\begin{aligned}
H_1(t) =& \sigma_S - A_0(t)S(t)\sigma_{\delta 1} + \sigma_{\delta 1}A_3(t)\sigma_{\kappa 1} - B_0(t)\sigma_{\xi 1} + \sigma_{\xi 1}B_3(t)\sigma_{\kappa 1},\\
H_2(t) =& - A_0(t)\sigma_{\delta 2} + \sigma_{\delta 2}A_5(t)\sigma_{\kappa 2} - B_0(t)\sigma_{\xi 2} + \sigma_{\xi 2}B_5(t)\sigma_{\kappa 2},\\
H_3(t) =& - A_0(t)\sigma_{\delta 3} + \sigma_{\delta 3}A_5(t)\sigma_{\kappa 3} - B_0(t)\sigma_{\xi 3} + \sigma_{\xi 3}B_7(t)\sigma_{\kappa 3},
\end{aligned}
$$

and $\sigma_{X1} - \sigma_{X3}$ and $\sigma_{Y1} - \sigma_{Y3}$ are the volatilities of the risky asset used to complete the market.

The proposition derived above is by using the method of replication based on Lioui [38, 39]. The stochastic dividend with stochastic earning yield is making an impact as similar to the model derived by Lioui. The delta shown in the proposition 5 can be either positive or negative, which is different to that of the Black-Scholes-Merton model. The reason for this is similar to what Lioui expressed in his paper [38]. The stock is hedged not only for the risk of the underlying, but also for the risk of the other parameters including the risk of dividend yield, the risk of market price of risk and the risk of earning yield. The sign of the call delta is time dependent, because the function $A_i(t)$, $B_i(t)$ and $D(t)$ are all time dependent, which implies that our method of portfolio replication might start with any position, either long or short position and end with any position as well and these affect the sign of the value.

The property of our proposition has more general settings than the Black-Scholes-Merton delta since it holds for stochastic parameters. However, it should be noted that for this proposition the delta is not calculated by the derivative of call option, $\frac{\partial C(t)}{\partial S(t)}$. By the derivative method, we obtain the following result.

Proposition 5.2. *The delta of the call option, Δ^*,with stochastic earning yield $\xi(t)$, stochastic dividend yield $\delta(t)$ and stochastic market price of risk $\kappa(t)$ written on $S(t)$ with exercise price K and maturity time T, is*

$$
\begin{aligned}
\Delta_C^*(t) &= \frac{\partial C(t)}{\partial S(t)}\\
&= e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)} N(d_1),
\end{aligned} \tag{5.17}
$$

where

$$
\begin{aligned}
d_1 =& d_2 + \Sigma(t), \\
d_2 =& \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r - \frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big), \\
\Sigma(t)^2 =& \int_t^T \varepsilon_1(s)^2 ds + \int_t^T \varepsilon_2(s)^2 ds + \int_t^T \varepsilon_3(s)^2 ds, \\
\Phi(t) =& A_0(t)\delta(t) + A_1(t) + B_0(t)\xi(t) + B_1(t) + D(t) \\
& - \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) - \sigma_{\delta_2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big) \\
& - \sigma_{\delta_3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) - \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa_1(t)\Big) \\
& - \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa_2(t)\Big) - \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa_3(t)\Big),
\end{aligned}
$$

and $A_0(t) - A_7(t)$*,* $B_0(t) - B_7(t)$*,* $D(t)$ *and* $\varepsilon_1(s) - \varepsilon_3(s)$ *are precisely given in the proposition 1.*

The delta is positive as expected. These results have more general settings than previous works [38, 77] as the model contains the stochastic earning yield.

5.3 Gamma

Gamma is the Greek parameter to measure the rate of change of the delta with respect to change in the underlying asset or the stock price. On the other words, Gamma is the second derivative of the option price with respect to the underlying asset value. Note that the gamma can be either positive or negative, and depends on what type of the options. Long options yield positive value while short options yield negative Gamma. Gamma has the significant property to correct the convexity of the value. For example, the investor might want to have the neutralized gamma of portfolio to assure that their ivestment is effective with any movement of the financial risky market. [78]

With the new parameters considered, including the stochastic dividend yield $\delta(t)$, the stochastic market price of risk $\kappa(t)$ and the stochastic earning yield $\xi(t)$, the extended option pricing model has been derived, and the gamma is obtained as follows.

Proposition 5.3. *The gamma of call option, $\Gamma(t)$, with stochastic earning yield $\xi(t)$, stochastic dividend yield $\delta(t)$ and stochastic market price of risk $\kappa(t)$ as the units number of the underlying asset in such a portfolio to replicate the call option written on $S(t)$ with exercise price K and maturity time T, is*

$$\begin{aligned}\Gamma(t) =& e^{\left(-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right)}\frac{N(d_1)}{S(t)\Sigma(t)^2}\cdot\frac{1}{\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2}}\\ &\times\Big[(\sigma_{X2}\sigma_{Y3}-\sigma_{X3}\sigma_{Y2})H_1(t)\\ &+(\sigma_{Y1}\sigma_{X3}-\sigma_{X1}\sigma_{Y3})H_2(t)\\ &+(\sigma_{X1}\sigma_{Y2}-\sigma_{Y3}\sigma_{X2})H_3(t)\Big],\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}H_1(t) =& \sigma_S - A_0(t)S(t)\sigma_{\delta1}+\sigma_{\delta1}A_3(t)\sigma_{\kappa1}-B_0(t)\sigma_{\xi1}+\sigma_{\xi1}B_3(t)\sigma_{\kappa1},\\ H_2(t) =& -A_0(t)\sigma_{\delta2}+\sigma_{\delta2}A_5(t)\sigma_{\kappa2}-B_0(t)\sigma_{\xi2}+\sigma_{\xi2}B_5(t)\sigma_{\kappa2},\\ H_3(t) =& -A_0(t)\sigma_{\delta3}+\sigma_{\delta3}A_5(t)\sigma_{\kappa3}-B_0(t)\sigma_{\xi3}+\sigma_{\xi3}B_7(t)\sigma_{\kappa3},\end{aligned}$$

and $\sigma_{X1}-\sigma_{X3}$ and $\sigma_{Y1}-\sigma_{Y3}$ are the volatilities of the risky asset used to complete the market.

The property of this gamma is the same as the delta in the previous section. The sign can be both positive and negative.

5.4 Vega

Vega is the Greek parameter to measure the sensitivity of the option value to the change of volatility. It is defined by the derivative of the option price with respect to the underlying stock volatility. The Greek parameter is one of the important measures to show the profit that the investor may receive when the volatility changes. Especially, for investment in the risky market, the option invested in the market is particularly sensitive to the change of market volatility. [78]

For the extended option pricing model taking into account the stochastic dividend yield $\delta(t)$, the stochastic market price of risk $\kappa(t)$ and the stochastic earning yield $\xi(t)$, based on equation (4.51) in proposition 4.2, the Greek parameter Vega is coumputed by differentiating $C(t)$ with respect to σ_S and the result is given by proposition 5.4 below.

Proposition 5.4. *The vega of call option, $\nu(t)$, with stochastic earning yield $\xi(t)$, stochastic dividend yield $\delta(t)$ and stochastic market price of risk $\kappa(t)$ as the units number of the underlying asset in such a portfolio to replicate the call option written on $S(t)$ with exercise price K and maturity time T, is*

$$
\begin{aligned}
\nu(t) =& S(t)e^{\left[-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right]}N(d_1)\frac{\sigma_S(T-t)}{\Sigma(t)} \\
&- S(t)e^{\left[-\frac{\sigma_S^2}{2}(T-t)-\Phi(t)+\frac{1}{2}\Sigma(t)^2\right]}N(d_1)\Big(\frac{1}{\Sigma(t)}+1\Big)\Psi(t),
\end{aligned}
\tag{5.19}
$$

where

$$
\begin{aligned}
d_1 =& d_2 + \Sigma(t), \\
d_2 =& \frac{1}{\Sigma(t)}\Big(ln\frac{S(t)}{K} + (r-\frac{\sigma_s^2}{2})(T-t) - \Phi(t)\Big), \\
\Sigma(t)^2 =& \int_t^T \varepsilon_1(s)^2 ds + \int_t^T \varepsilon_2(s)^2 ds + \int_t^T \varepsilon_3(s)^2 ds, \\
\Phi(t) =& A_0(t)\delta(t) + A_1(t) + B_0(t)\xi(t) + B_1(t) + D(t) \\
&- \sigma_{\delta 1}\Big(A_2(t) - A_3(t)\kappa_1(t)\Big) - \sigma_{\delta_2}\Big(A_4(t) - A_5(t)\kappa_2(t)\Big) \\
&- \sigma_{\delta_3}\Big(A_6(t) - A_7(t)\kappa_3(t)\Big) - \sigma_{\xi 1}\Big(B_2(t) - B_3(t)\kappa_1(t)\Big) \\
&- \sigma_{\xi 2}\Big(B_4(t) - B_5(t)\kappa_2(t)\Big) - \sigma_{\xi 3}\Big(B_6(t) - B_7(t)\kappa_3(t)\Big), \\
\Psi(t) =& -\int_t^T (\varepsilon_1(s) - \sigma_S)ds,
\end{aligned}
$$

and $A_0(t) - A_7(t)$, $B_0(t) - B_7(t)$, $D(t)$ and $\varepsilon_1(s) - \varepsilon_3(s)$ are precisely given in the proposition 1.

The result of the vega that we obtained is different from the Black-Scholes-Merton model which only provides a positive value. The model of our work with the existence of the stochastic earning yield can be either positive or negative which is similar to Loui's work with the presence of the stochastic dividend yield (2006) [38] and the work of Karoui, Jeanblanc-Picque and Shreve with the presence of the stochastic volatility (1998) [79]. Our work complements the previous works.

In summary, our option pricing model yields a stochastic value of option price because of the presence of three stochastic parameters, including the stochastic earning yield, the stochastic dividend yield and the stochastic market price of risk. The option prices are affected by these three stochastic parameters.

5.5 Concluding Remarks

The Greek parameters are the measures of the sensitivity of the option value to the change of the model parameters. Computation of these parameters are important as they help to manage the risk to hedge the portfolio. In the study of option pricing models, the Greek parameters are usually analyzed as the testing of the models.

In this chapter, we successfully establish analytical results for the Greek parameters delta $\Delta(t)$, gamma $\Gamma(t)$, and vega $\nu(t)$. All the results are for call option. The Greek parameter delta is determined in two methods, as the units number of the underlying in the portfolio which is replicating to the option value and as the derivative of the option price with respect to the stock price; while the Greek parameter gamma and vega are determined respectively as the derivative of the delta with respect to the stock price, and the derivative of the option price with respect to the stock volatility.

The results obtained possess very interesting stochastic features which is due to the consideration of the stochastic characteristic of the model parameters. Our results show some similarities to those by the option pricing model with the stochastic volatility [79] and the model with stochastic dividend yield [38]. Our results complements previous works by including the stochastic earning yield in the option pricing model.

Chapter 6

Performance of Option Pricing Model with Stochastic Earning Yield

6.1 General

In Chapters 4 and 5, we develop an extended option pricing model taking into account the stochastic earning yield and establish analytical results for the Greek parameters. In this chapter, we will investigate the effectiveness of the model with particular focus on tackling the following issues/questions: Is the model applicable? What is the implications from stochastic earning yield for the option properties? What is the sensitivity of the option price to the model parameters? Does the model perform a suitable property of option price? There are several parameters in our new option pricing model, including strike time T, the volatility of the underlying asset σ_S, the volatility of the dividend yield σ_{δ_i}, the volatility of the earning yield σ_{ξ_i}, the volatility of the market price of risk σ_{κ_i}, the dividend yield friction coefficient θ_δ, the earning yield friction coefficient θ_ξ and the market price of risk friction coefficient θ_{κ_i}. In this chapter we will test the performance of the model by checking the sensitivity of the option pricing model to each of the key parameters.

Table 6.1: Parameter values used in the sensitivity analysis simulation

$K = 500$	$\delta(0) = 0.05$	$\xi(0) = 0.05$	$\kappa_1(0) = 0.20$
$r = 0.05$	$\mu_\delta = 0.05$	$\mu_\xi = 0.05$	$\kappa_2(0) = 0.20$
			$\kappa_3(0) = 0.20$
			$\mu_{\kappa_1} = 0.20$
			$\mu_{\kappa_2} = 0.20$
			$\mu_{\kappa_3} = 0.20$

6.2 Sensitivity Analysis

Sensitivity Analysis is an approach to analyze the model by observing the changes of the model output while changing the model inputs. This analysis is an important investigation for the development of new models, especially for developing models to be used for decision making and for stochastic models with uncertainty.

As our extended model includes various parameters, the sensitivity analysis is to be carried out to investigate the importance and the level of impact of each of the parameters. The parameters to be investigated include time maturity T, the volatility of the underlying asset σ_S, the volatility of the dividend yield σ_{δ_i}, the volatility of the earning yield σ_{ξ_i}, the volatility of the market price of risk σ_{κ_i}, the dividend yield friction coefficient θ_δ, the earning yield friction coefficient θ_ξ and the market price of risk friction coefficient θ_{κ_i}.

In order to test the sensitivity of the option price to the change of parameter values, in this section, we perform some simulations by globally setting the parameters as shown in the table 6.1,which is the same as Abraham Lioui's article [38]. The analysis of sensitivity will be described by two aspects: the influence of the parameter itself and the variation due to the use of different models. The first aspect is to observe the trend of sensitivity as the parameter changes, while the second perspective is to study the pricing property of the models.

In this section, we will analyze the sensitivity of our model by simulating the variation of the option prices in response to the change of the parameter in question. Also we will compare the option prices determined by the deterministic dividend yield Black-Scholes-Merton model with the option prices obtained from our stochastic earning yield option pricing model.

6.2.1 Dividend Yield Friction Coefficient (θ_δ)

In the Ornstein-Uhlenbeck process, the θ is the coefficient of friction. One of the well known applications of the Ornstein-Uhlenbeck process is a noisy relaxation process prototype. A Hookean spring is an example in which change of the friction coefficient may lead to qualitative change of the dynamics of the system [73]. Friction is the resisting force which impacts on the motion of an object. There are many types of friction in physics, such as dry friction, fluid friction, skin friction and internal friction.

In our GOU process which is applied to the financial market, the coefficient of the process is treated as the financial market friction. The obvious examples for the financial market friction are taxes and transactions costs. In fact, market frictions can be anything which can affect the financial market. Especially for the stock market, the taxes of capital gains have a significant influence on investors' decision for trading the stocks or hedging the options. The stock market friction can be a non-monetary factor which interferes with trading market. This interference includes two dimensions, three dimensions, or n dimensions. Stock market frictions cause the investors to deviate from holding the underlying portfolio. This implies that frictions can make a significant impact on the company or even the whole financial market. [80]

The dividend yield friction coefficient θ_δ is the friction factor which may cause the change of dividend yield process. In this section we will analyze the sensitivity of the option prices to this parameter. We test the influence of the dividend yield friction coefficient by changing this parameter at three values: $\theta_\delta(DYF) = 0.20$, $\theta_\delta(DYF) = 0.50$ and $\theta_\delta(DYF) = 0.90$, while holding the other parameters at the values as shown in the table 6.2. The results of simulations are shown in the Figure 6.1-6.2.

From Figure 6.1-6.2, we can see that the dividend yield friction coefficient has significant impact on the option prices, no matter when the option is in-the-

Table 6.2: Parameter values used in the sensitivity analysis of the model with respect to the dividend yield friction coefficient

$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\theta_\xi = 0.25$	$\theta_{\kappa 1} = 0.45$
$\sigma_S = 0.20$	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\theta_{\kappa 3} = 0.45$
		$\sigma_{\xi 3} = 0.05$	$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$

money or out-of-the-money. Especially for the dividend yield friction coefficient of 0.50, the price of call option increases significantly and the price of put option decreases drastically as the moneyness increases.

In figure 6.3, the zoom of the picture 6.2b is shown. The graph of the dividend yield friction coefficient of 0.50 is totally different to those for the other two coefficient values. This implies that the dividend yield friction coefficient can be interpreted as the situation of the financial market that affects the dividend yield process.

In summary, the dividend yield friction coefficient significantly affects the sensitivity of option prices for both types of options since it reflects the situation of the financial market. Consequently, in the next section, we should test the sensitivity in each situation to see the difference of sensitivity results.

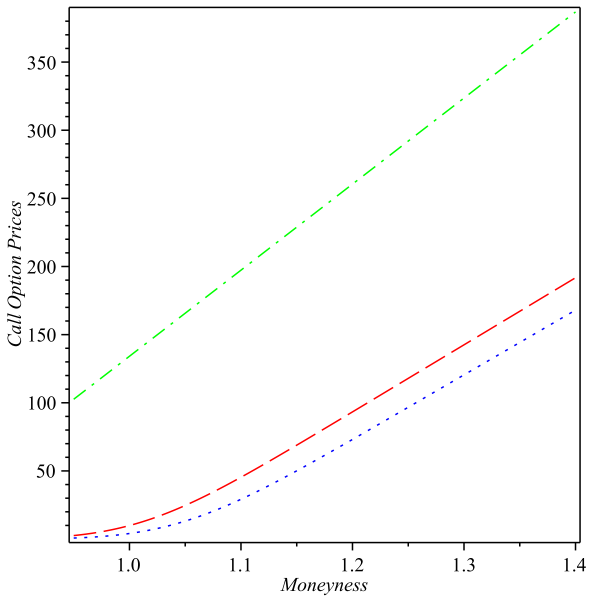
6.2.2 Earning yield friction coefficient (θ_ξ)

Since the earning yield process is assumed by the GOU process, the θ_ξ is the earning yield friction coefficient which has the similar property as the dividend yield friction coefficient. The difference is that the earning yield friction coefficient determines the situation of friction which has influence on the earning yield process, for example the factor that can make an impact to the income statement of a company.

Table 6.3: Parameter values used in the sensitivity analysis of the model with respect to the earning yield friction coefficient

$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\theta_{\kappa 1} = 0.45$
$\sigma_S = 0.20$	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 3} = 0.05$	$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$

In this section, we analyze the sensitivity of option prices to the earning yield friction coefficient by changing its value. We change the earning yield friction coefficient at three values, $\theta_\xi = 0.20, 0.50$ and 0.90, while holding other parameters unchanged at the values gives in table 6.3. As shown in the previous section, dividend yield friction coefficient can create different situation of the stock market, and thus we will analyze the influence of the earning yield friction coefficient under three situations ($\theta_\delta = 0.35$, $\theta_\delta = 0.65$ and $\theta_\delta = 0.95$). The simulation results are shown in figures 6.4-6.11.



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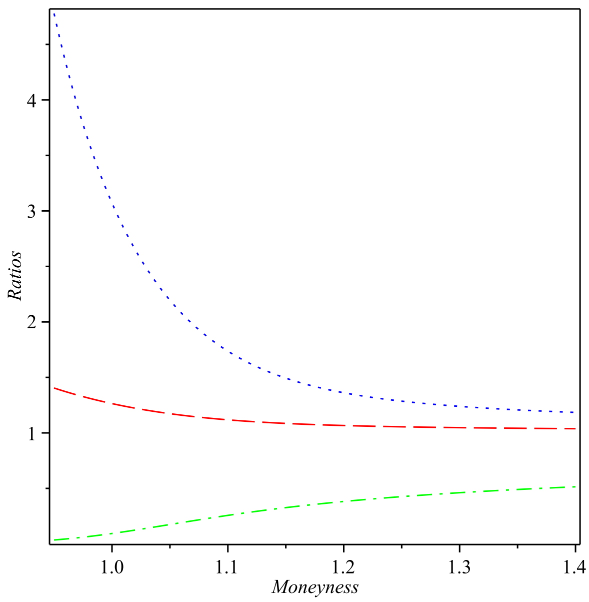
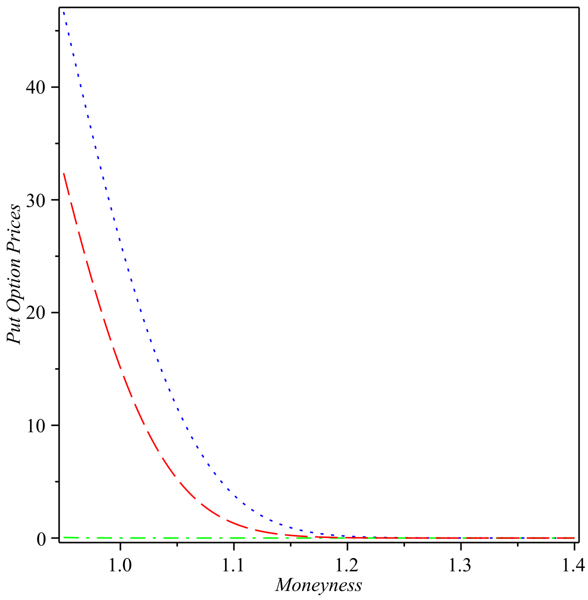


(b)

Figure 6.1: Sensitivity of call option prices to the dividend yield friction coefficient θ_δ. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\delta = 0.20$ (dotted line), $\theta_\delta = 0.50$ (dash-dotted line), $\theta_\delta = 0.90$ (dashed line).



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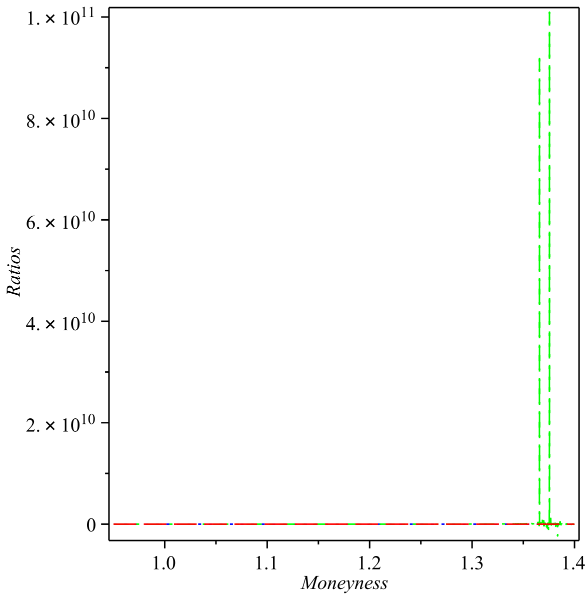
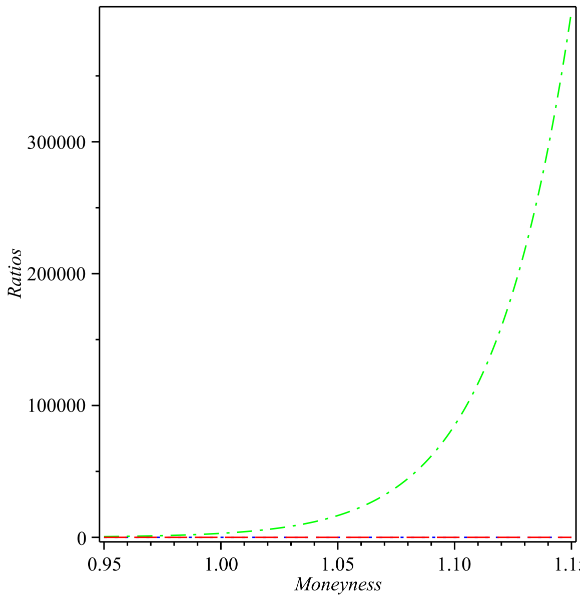


(b)

Figure 6.2: Sensitivity of put option prices to the dividend yield friction coefficient θ_δ. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\delta = 0.20$ (dotted line), $\theta_\delta = 0.50$ (dash-dotted line), $\theta_\delta = 0.90$ (dashed line).



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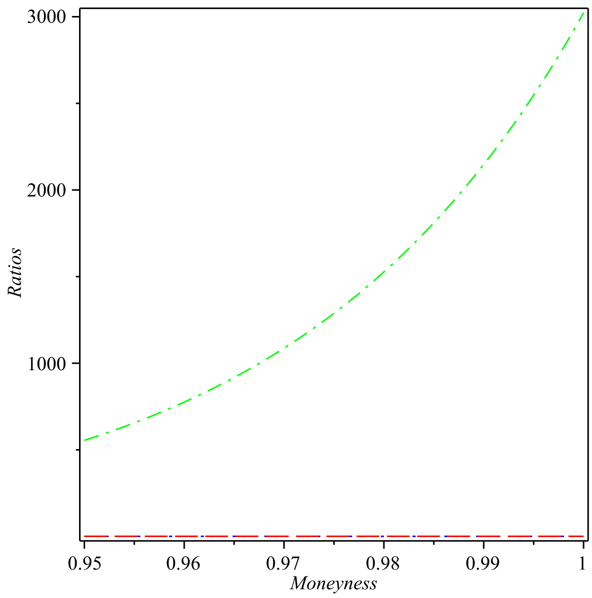
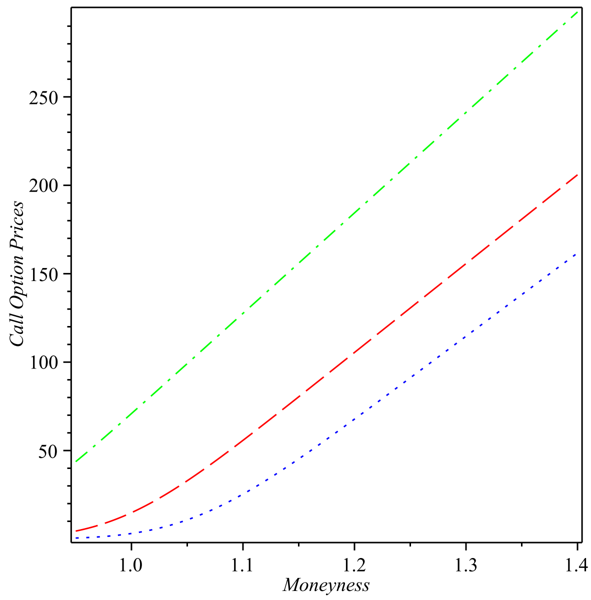


(b)

Figure 6.3: Range zoom of the figure (6.2b) for the sensitivity of put option prices to the dividend yield friction coefficient θ_δ. In the figure, $\theta_\delta = 0.20$ (dotted line), $\theta_\delta = 0.50$ (dash-dotted line), $\theta_\delta = 0.90$ (dashed line).



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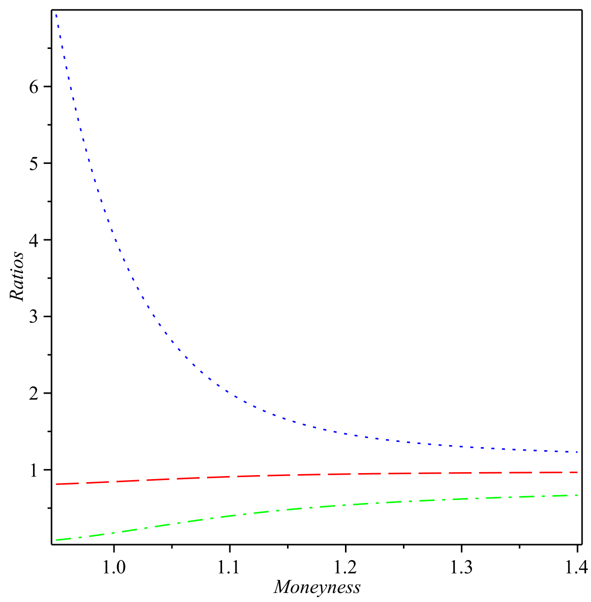
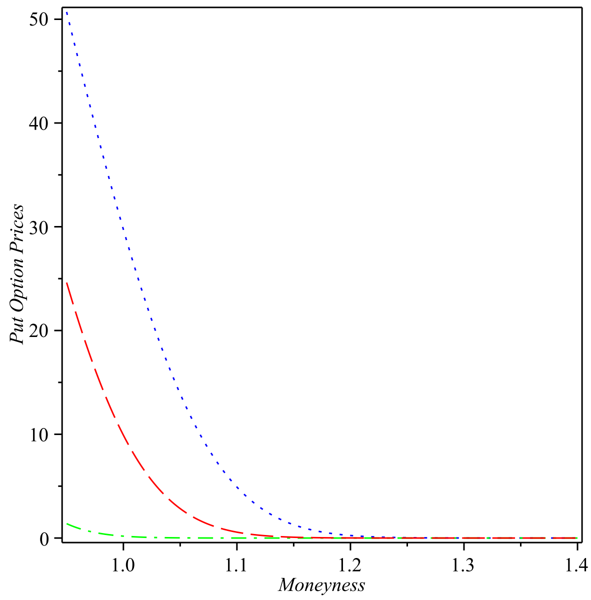


(b)

Figure 6.4: Sensitivity of call option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.35$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = 0.20$ (dotted line), $\theta_\xi = 0.50$ (dash-dotted line), $\theta_\xi = 0.90$ (dashed line).



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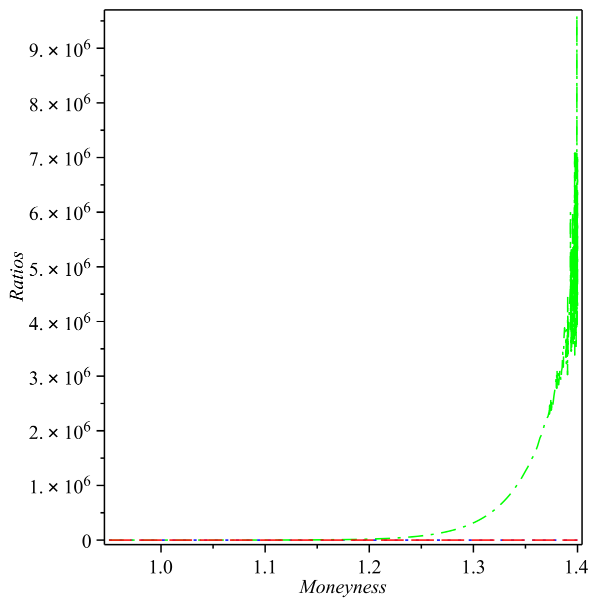
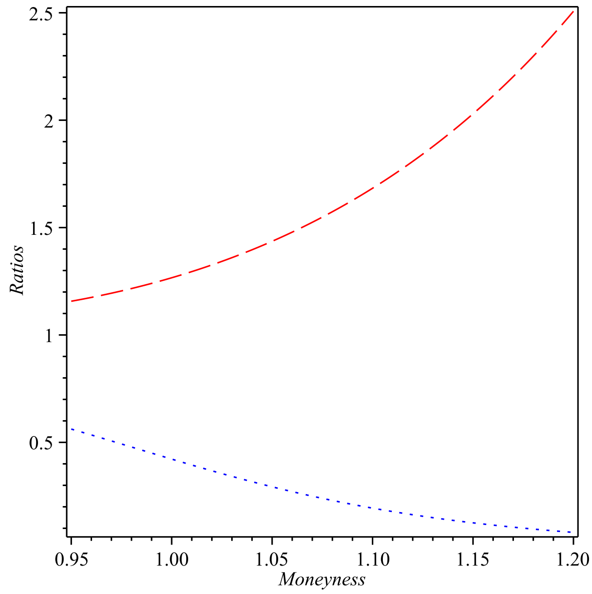


(b)

Figure 6.5: Sensitivity of put option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.35$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = 0.20$ (dotted line), $\theta_\xi = 0.50$ (dash-dotted line), $\theta_\xi = 0.90$ (dashed line).



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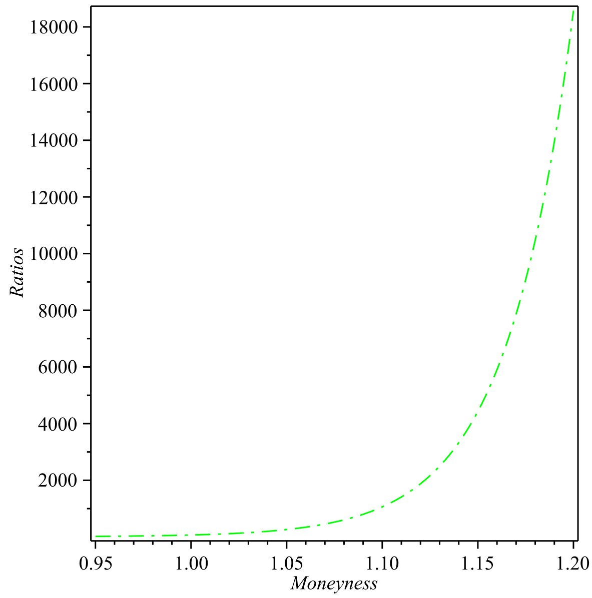

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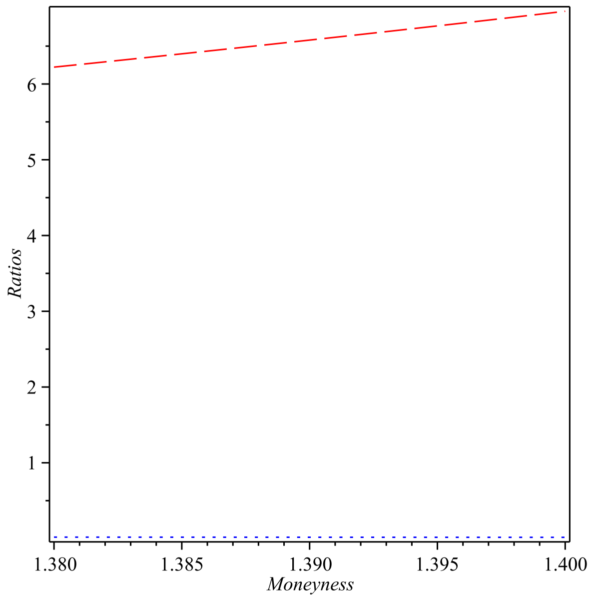
Figure 6.6: Range zoom of the figure (6.5b) for the sensitivity of put option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.35$. In the figure, $\theta_\delta = 0.20$ (dotted line), $\theta_\delta = 0.50$ (dash-dotted line), $\theta_\delta = 0.90$ (dashed line).



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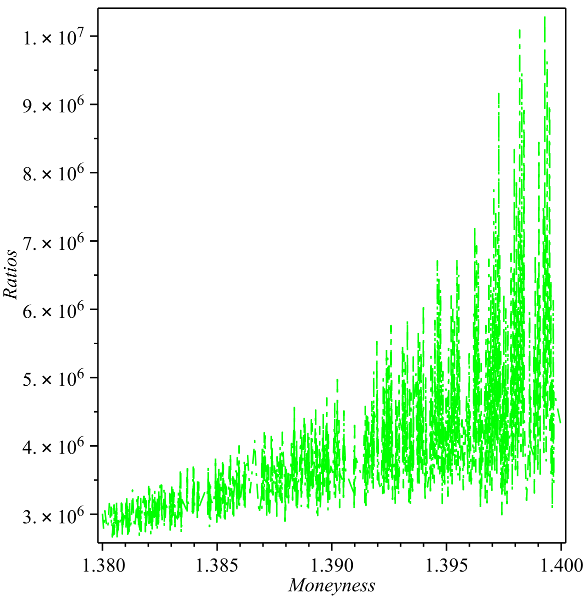

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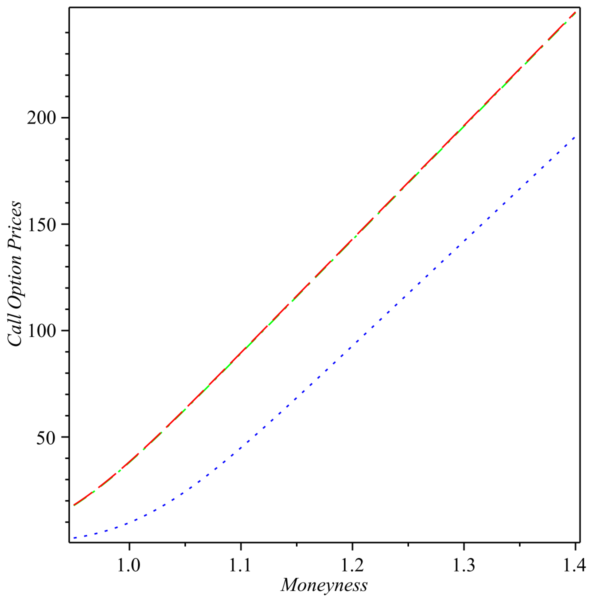
Figure 6.7: Range zoom of the figure (6.5b) for the sensitivity of put option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.35$. In the figure, $\theta_\delta = 0.20$ (dotted line), $\theta_\delta = 0.50$ (dash-dotted line), $\theta_\delta = 0.90$ (dashed line).



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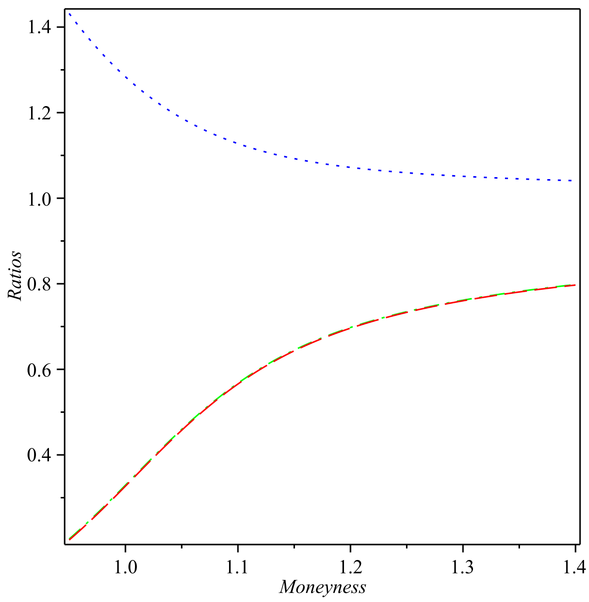
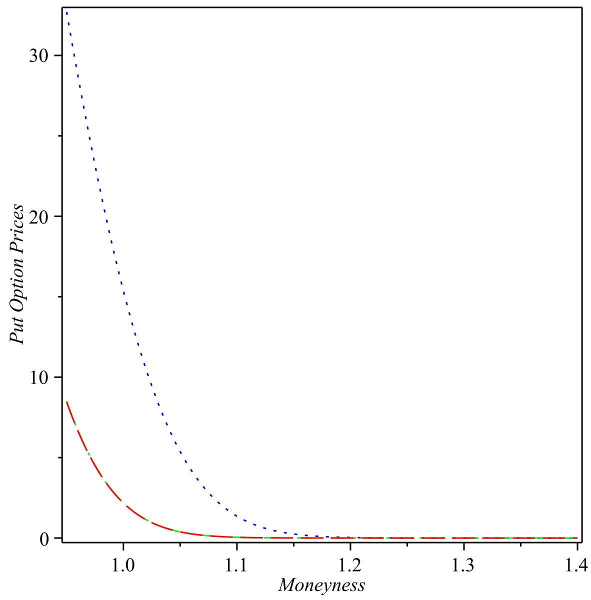


(b)

Figure 6.8: Sensitivity of call option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.65$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = 0.20$ (dotted line), $\theta_\xi = 0.50$ (dash-dotted line), $\theta_\xi = 0.90$ (dashed line).



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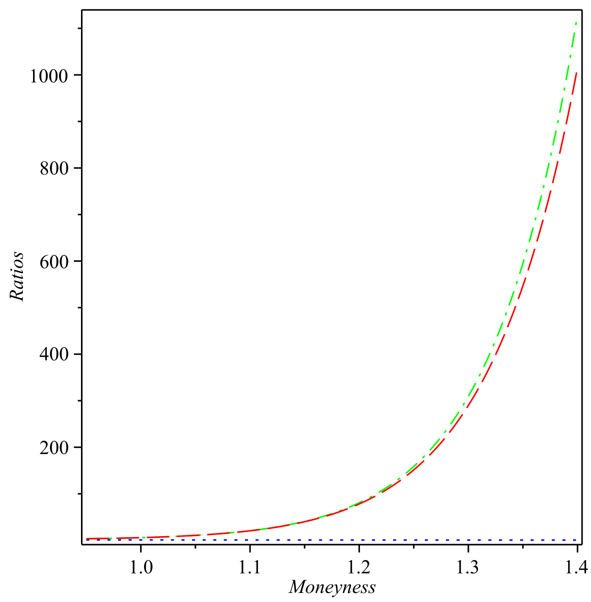
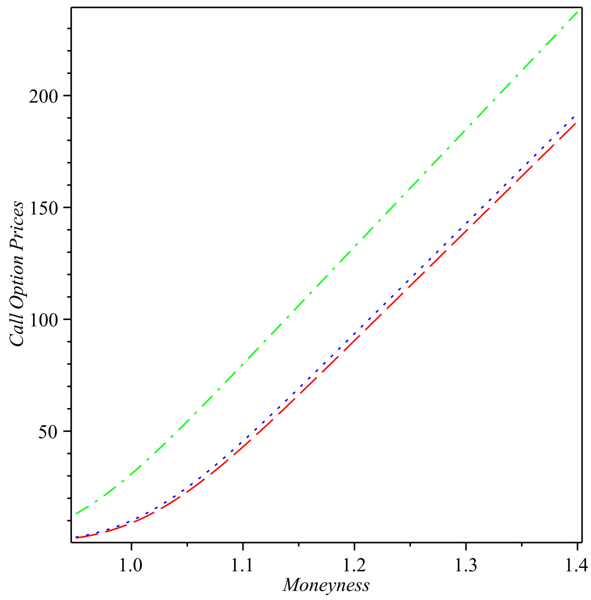


(b)

Figure 6.9: Sensitivity of put option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.65$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = 0.20$ (dotted line), $\theta_\xi = 0.50$ (dash-dotted line), $\theta_\xi = 0.90$ (dashed line).



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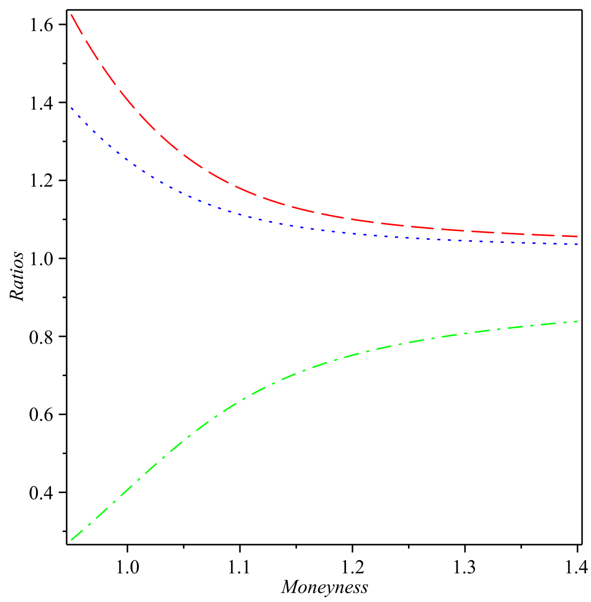
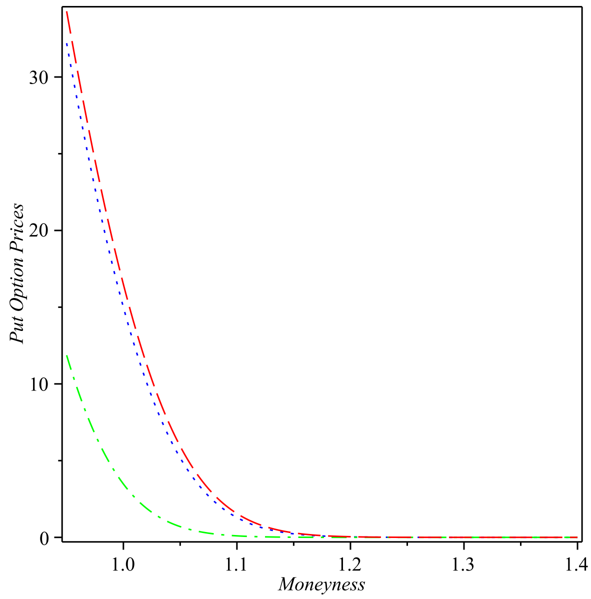


(b)

Figure 6.10: Sensitivity of call option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.95$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = 0.20$ (dotted line), $\theta_\xi = 0.50$ (dash-dotted line), $\theta_\xi = 0.90$ (dashed line).



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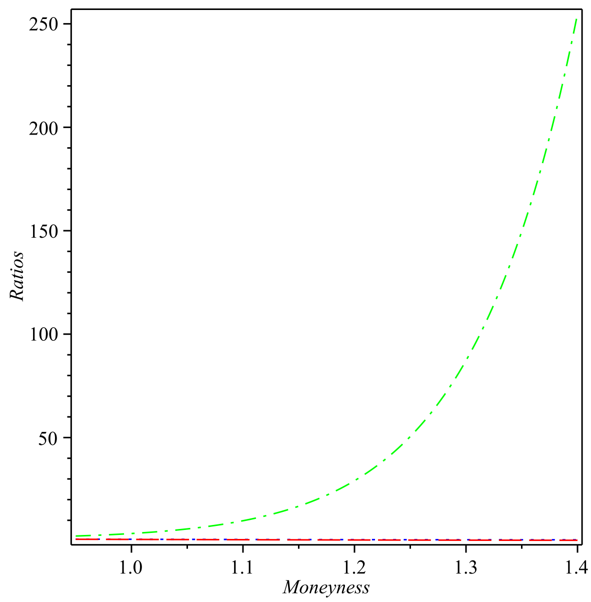
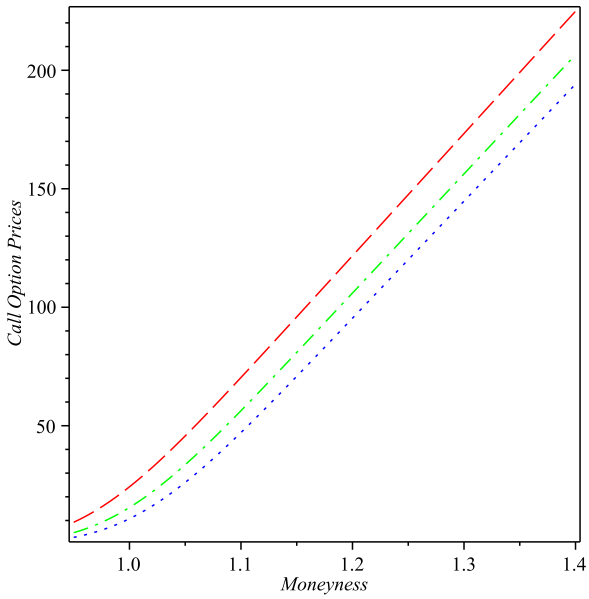


(b)

Figure 6.11: Sensitivity of put option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.95$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = 0.20$ (dotted line), $\theta_\xi = 0.50$ (dash-dotted line), $\theta_\xi = 0.90$ (dashed line).

The simulation results show that the earning yield friction coefficient can make a great impact on option prices for both call and put types. For the situation of $\theta_\delta = 0.35$, under environment of the earning yield friction coefficient at 0.50, the option price are noticeably different showing the strong stochastic behaviour. The call prices increase and the put prices decrease significantly as the moneyness increases. Figures 6.5-6.7 show that the parameter of earning yield friction coefficient has more impact to the out-of-the-money or deep-out-of-the-money options. Especially, figure 6.7 shows the uncertainty property when the option is very-deep-out-of-the-money. However when the situation is changed by setting $\theta_\delta = 0.65$, $\theta_\xi = 0.50$ has similar impact as $\theta_\xi = 0.90$, and still make more impact on the out-of-the-money or deep-out-of-the-money options. The co-relationship between θ_δ and θ_ξ has brought different kinds of situations and it wisely explains the situation of stochastic financial market in the real world.

From (4.8), if θ_ξ is taken as positive value, then $d\delta(t)$ will be positive when $\xi(t)$ increases to values greater than μ_ξ. However, by taking θ_ξ as negative value, then the model indicates that $\delta(t)$ increases when $\xi(t)$ is greater than the mean earning yield μ_ξ. Figures 6.12-6.13 show the influence of negative θ_ξ on the option price when $\theta_\delta = 0.35$. It is clear that θ_ξ can be any real number since the earning yield rate is driven independently and depends on the company policy. For the simplicity of our investigation of simulation, we will investigate the positive case of θ_ξ.



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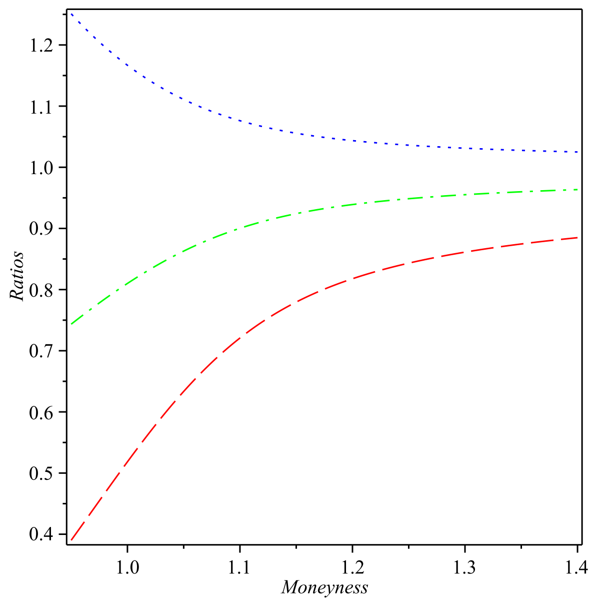
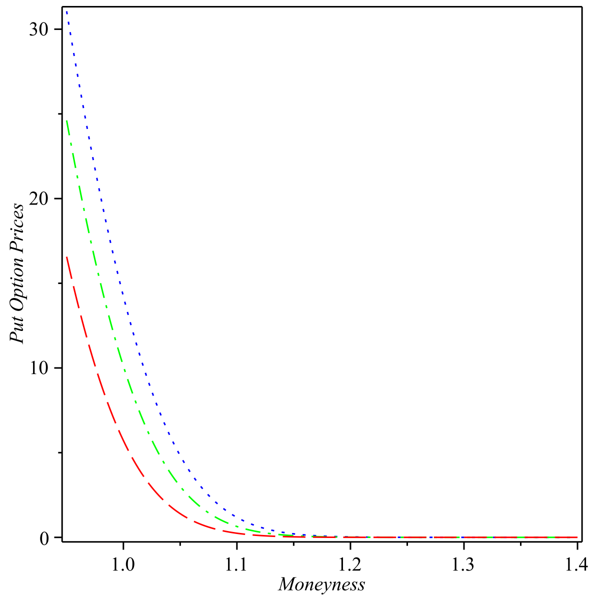


(b)

Figure 6.12: Sensitivity of call option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.35$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = -0.20$ (dotted line), $\theta_\xi = -0.50$ (dash-dotted line), $\theta_\xi = -0.90$ (dashed line).



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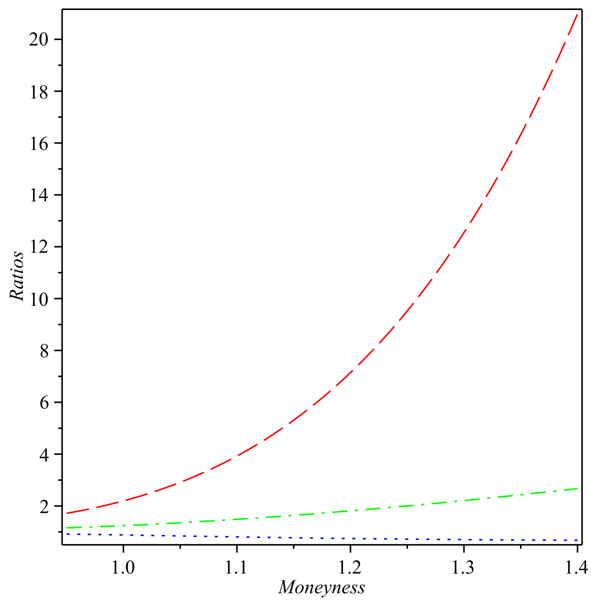


(b)

Figure 6.13: Sensitivity of put option prices to the earning yield friction coefficient θ_ξ for $\theta_\delta = 0.35$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_\xi = -0.20$ (dotted line), $\theta_\xi = -0.50$ (dash-dotted line), $\theta_\xi = -0.90$ (dashed line).

In the next section, we will consider different combinations of the dividend yield friction coefficient θ_δ and the earning yield friction coefficient θ_ξ to reflect the real situation of financial market since both parameters have huge impact and high sensitivity to the option prices.

6.2.3 Market Price of Risk Friction Coefficient (θ_{κ_i})

In this thesis, we study the influence of the market price of risk on the movement of the market. The market price of risk, κ_i, is determined as the market price of risk of each fundamental source of risk. And we define the process of this risk by the Ornstein-Uhlenbeck process, similar to Lioui's work [38], but we include the source or risk from the earning yield process. Hence, the market price of risk process includes the friction coefficient which can explain the situation of the market in each source of risk.

For the sensitivity analysis of this parameter, we follow the same approach as in the previous sections. Under different situations of the market, we set the model parameters at the values as shown in Table 6.4 but change the value of the market price of risk friction coefficient. In this work, we assume the market price of risk friction coefficient for all sources of risk i to have the same value, for example, if we set $\theta_{\kappa_i} = 0.20$, then it implies that we simulate under the circumstance of $\theta_{\kappa_1} = 0.20$, $\theta_{\kappa_2} = 0.20$ and $\theta_{\kappa_3} = 0.20$. By this approach, we can observe the sensitivity of option prices to this parameter, the market price of risk friction coefficient.

Table 6.4: Parameter values used in the sensitivity analysis of the model with respect to the market price of risk friction coefficient

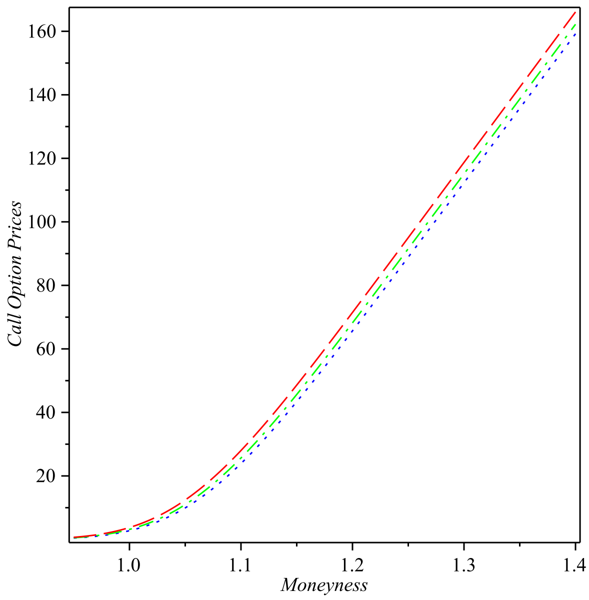
$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\sigma_{\kappa 1} = 0.10$
$\sigma_S = 0.20$	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\sigma_{\kappa 1} = 0.10$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 3} = 0.05$	$\sigma_{\kappa 1} = 0.10$

Under different combination of θ_δ and θ_ξ, we simulate the sensitivity of option prices to the market price of risk friction coefficient under the various market situations where θ_δ and θ_ξ are set to values as shown in the table 6.6.

Table 6.5: Market situation setting by the combination of dividend yield coefficient (θ_δ) and earning yield coefficient (θ_ξ)

θ_δ	θ_ξ
0.35	0.25
0.35	0.55
0.35	0.85
0.65	0.25
0.65	0.55
0.65	0.85
0.95	0.25
0.95	0.55
0.95	0.85

Figures 6.4-6.31 show the results of simulation for the sensitivity of the model with respect to the market price of risk friction coefficient.



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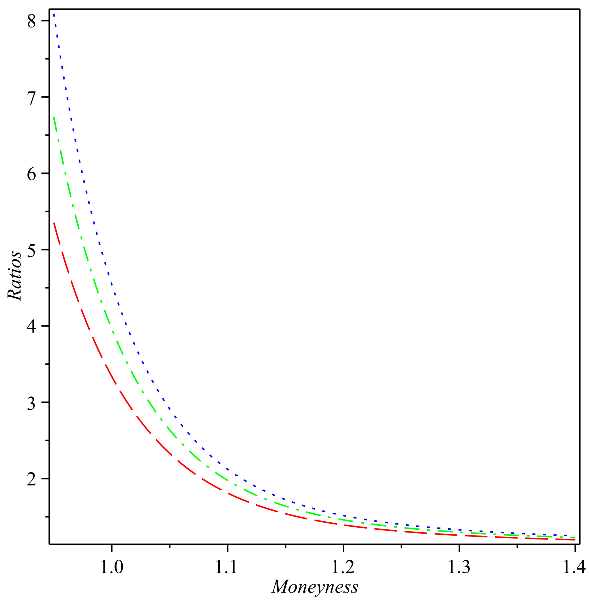
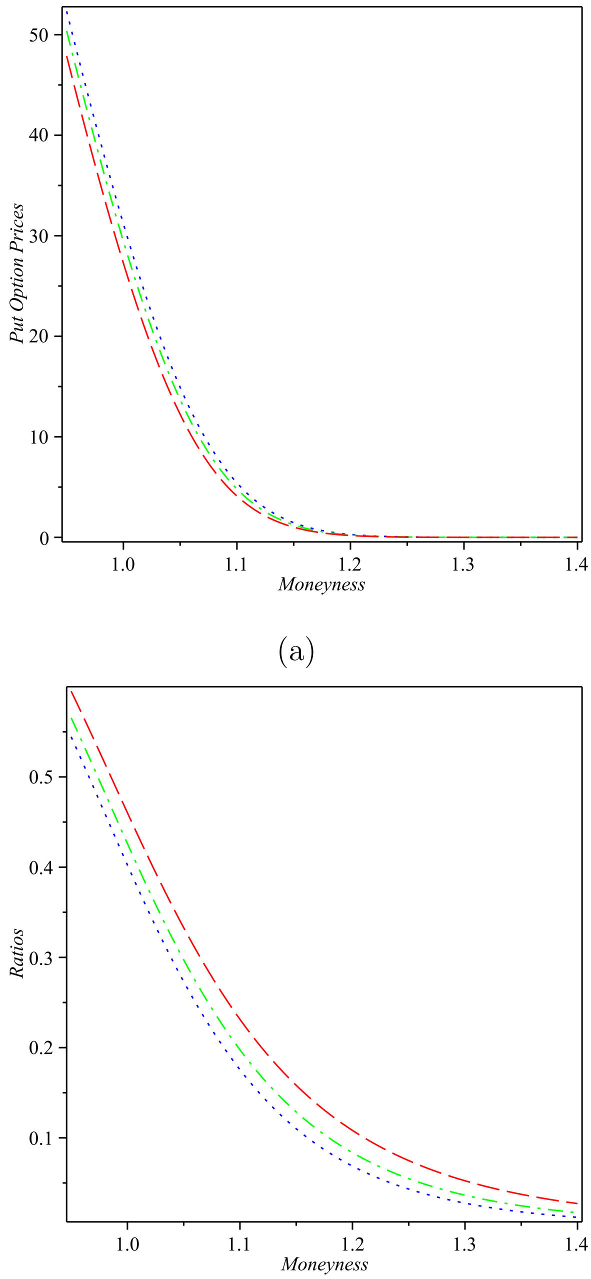


(b)

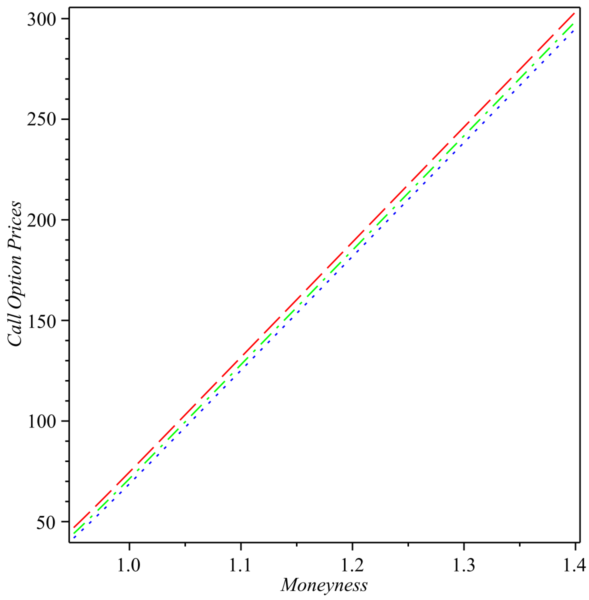
Figure 6.14: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



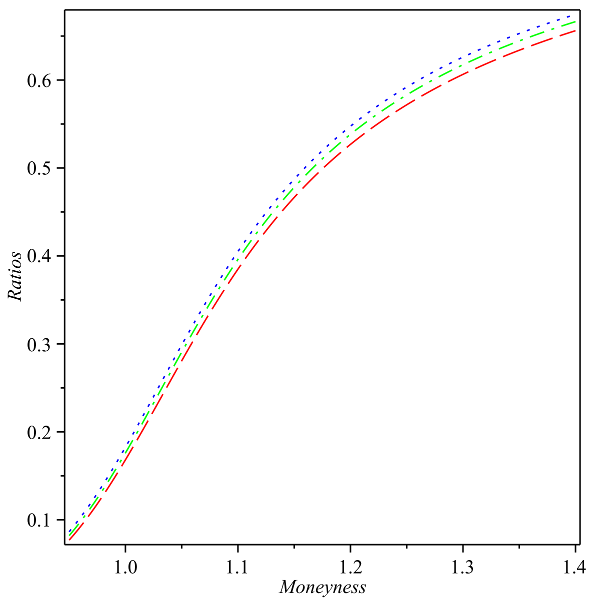
(a)

(b)

Figure 6.15: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).

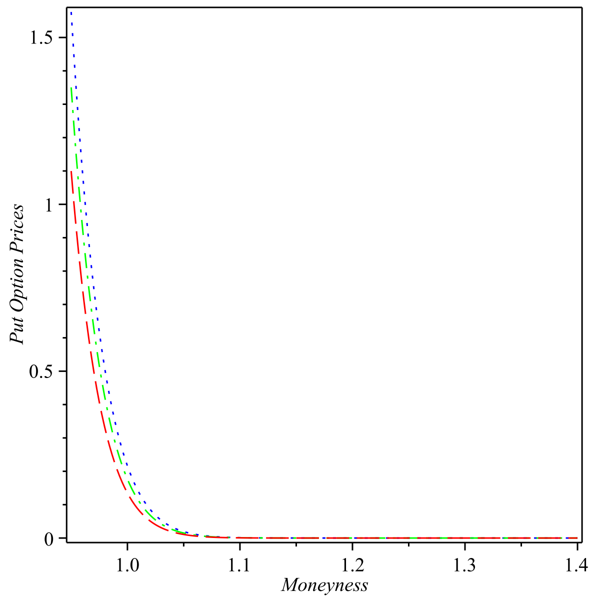


(a)



(b)

Figure 6.16: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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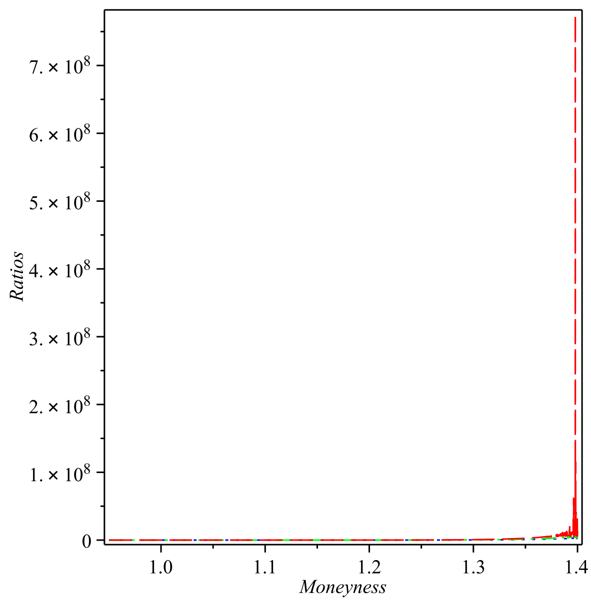
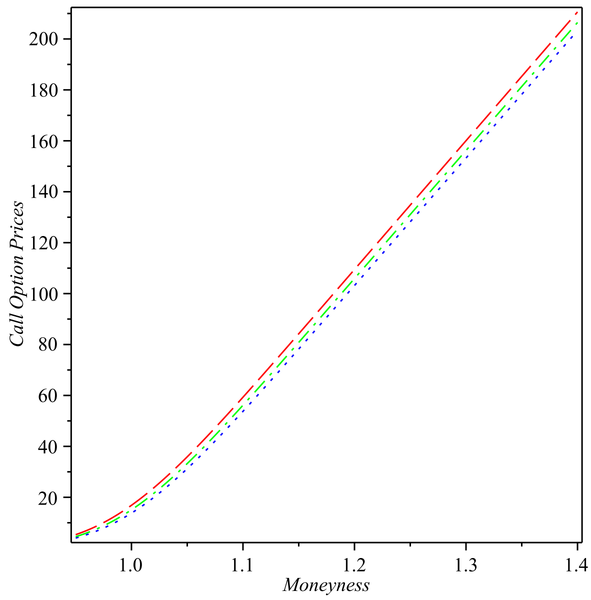


(b)

Figure 6.17: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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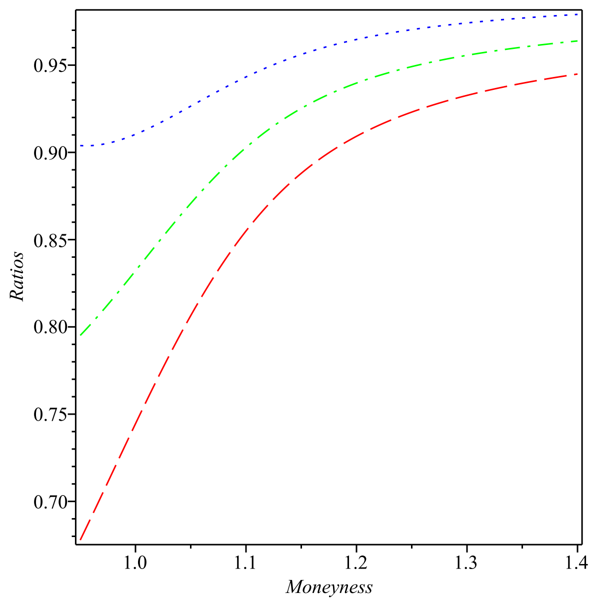
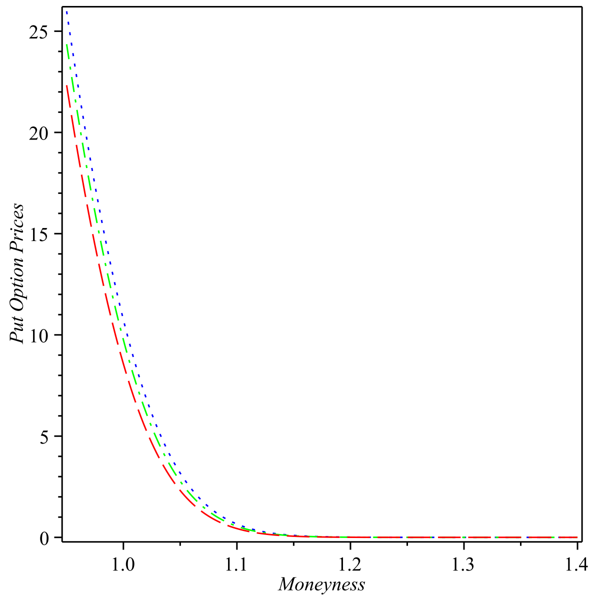


(b)

Figure 6.18: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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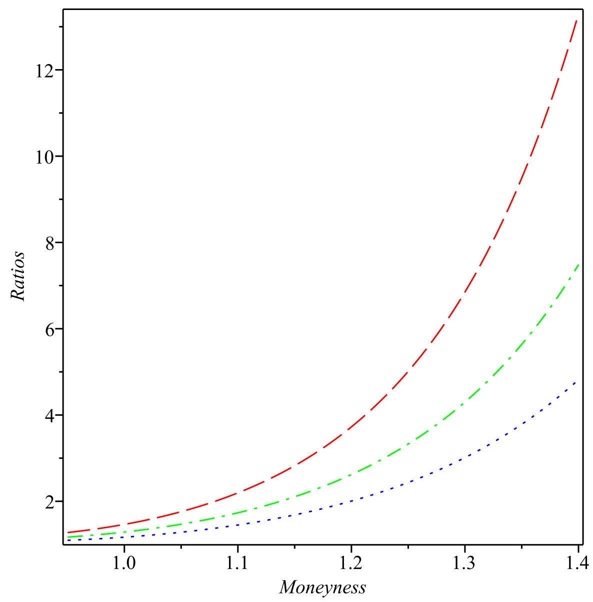
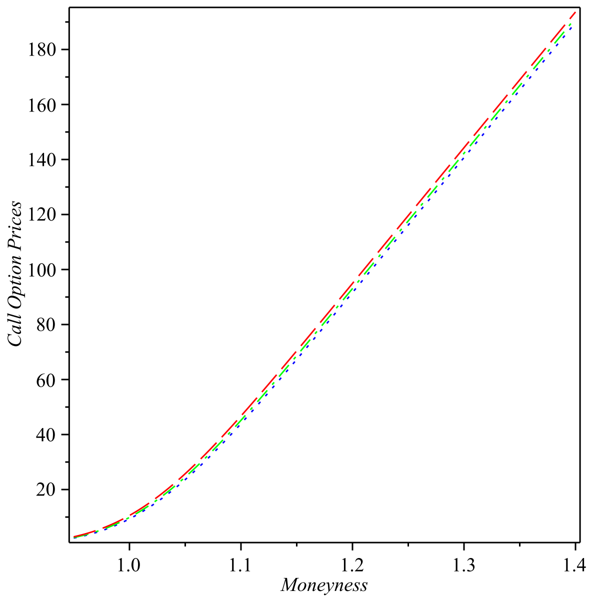


(b)

Figure 6.19: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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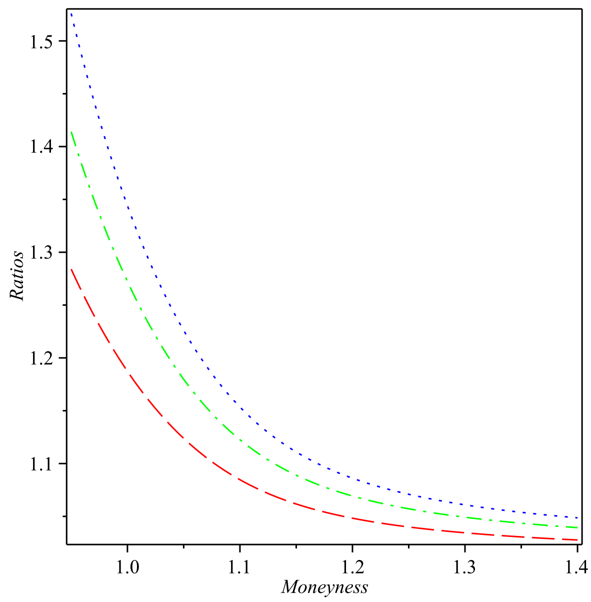
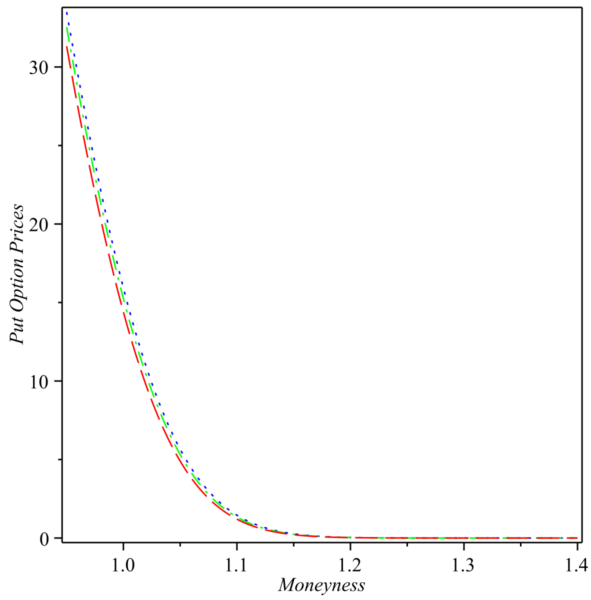


(b)

Figure 6.20: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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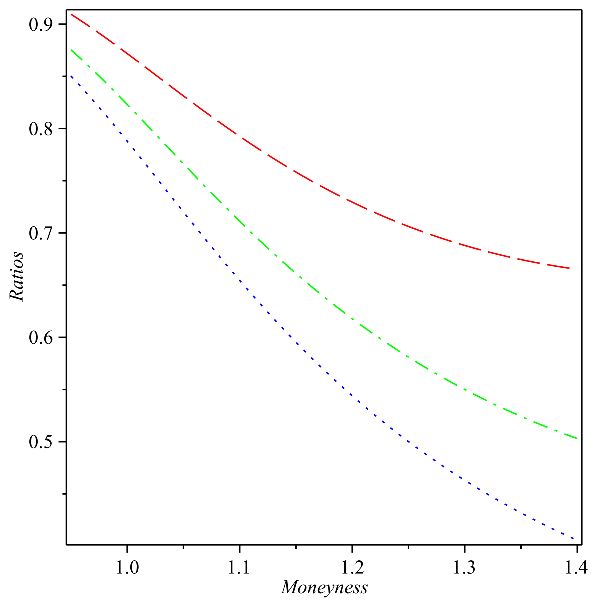
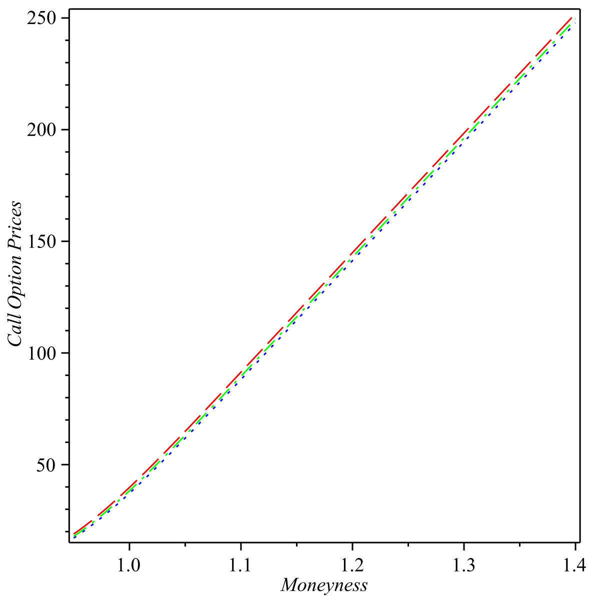


(b)

Figure 6.21: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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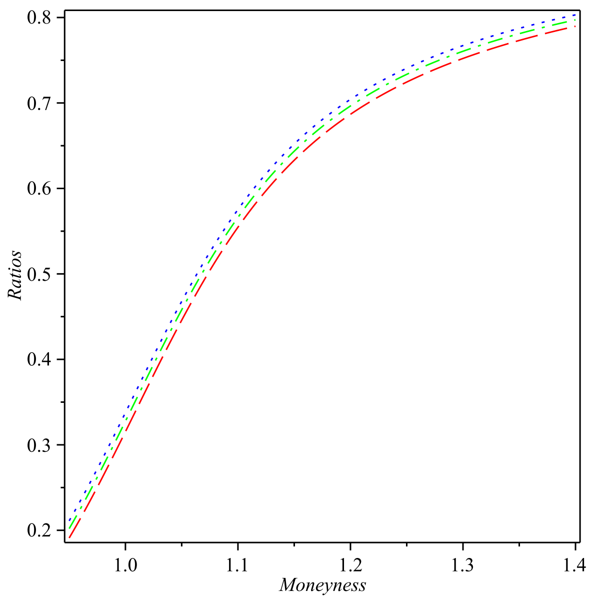
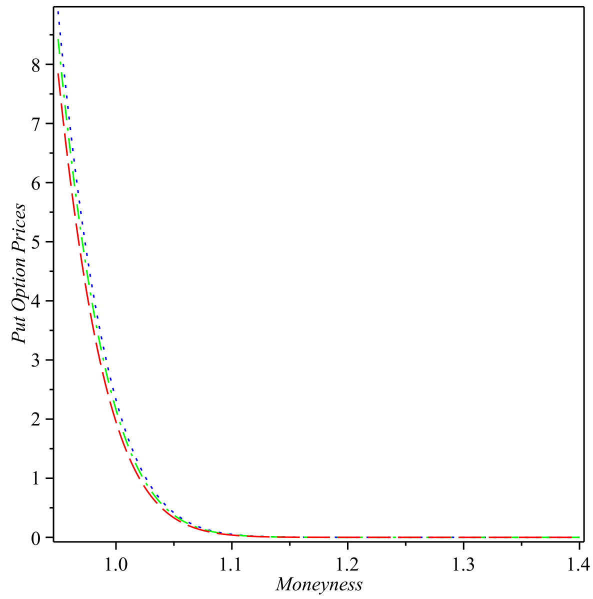


(b)

Figure 6.22: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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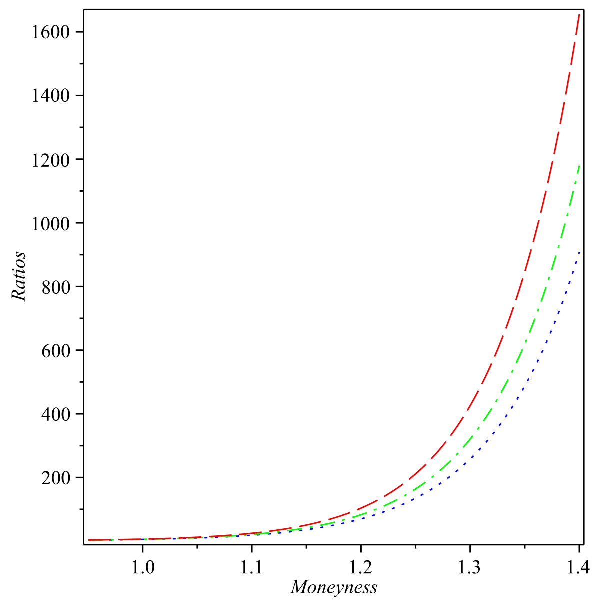
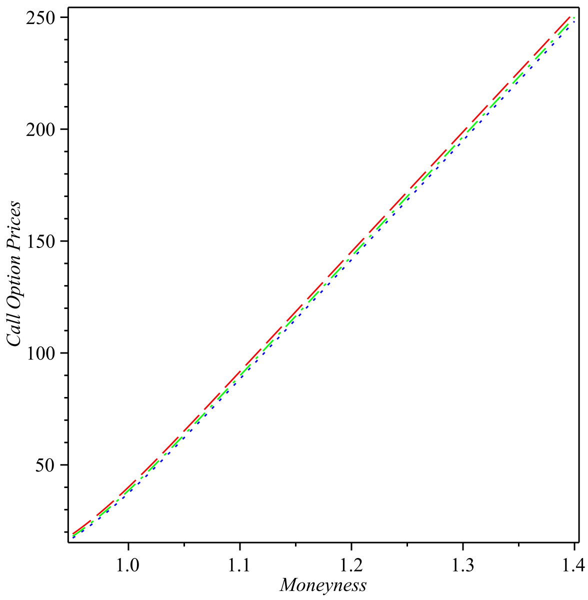


(b)

Figure 6.23: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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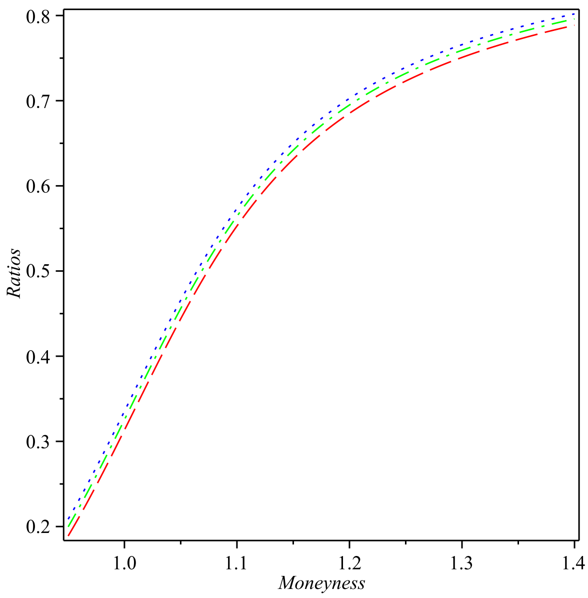
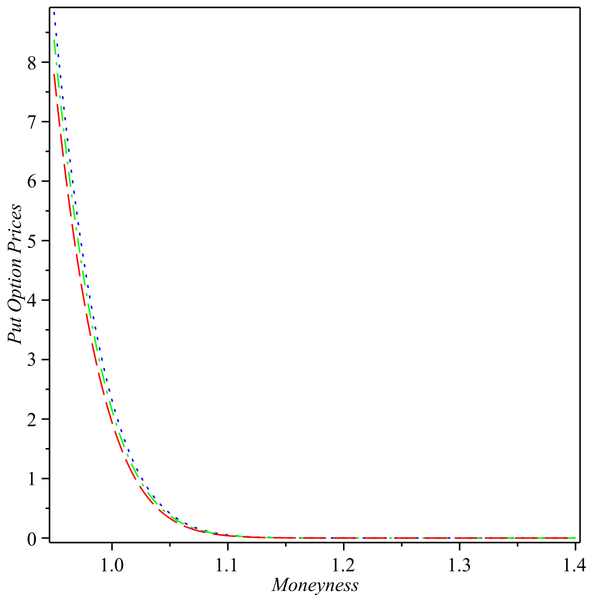


(b)

Figure 6.24: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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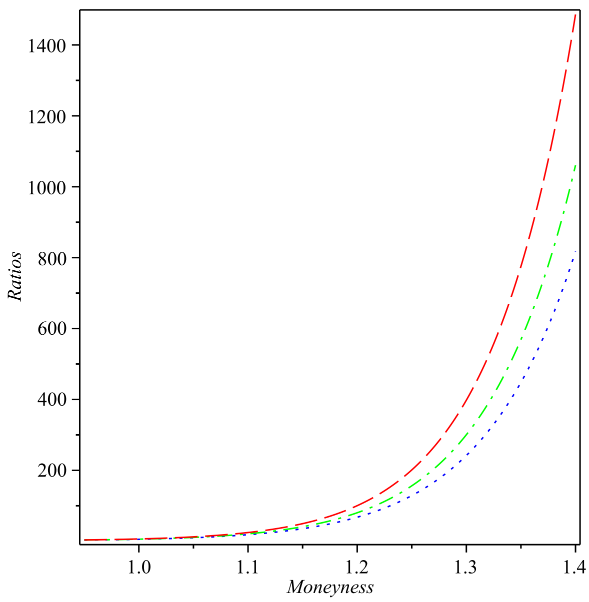
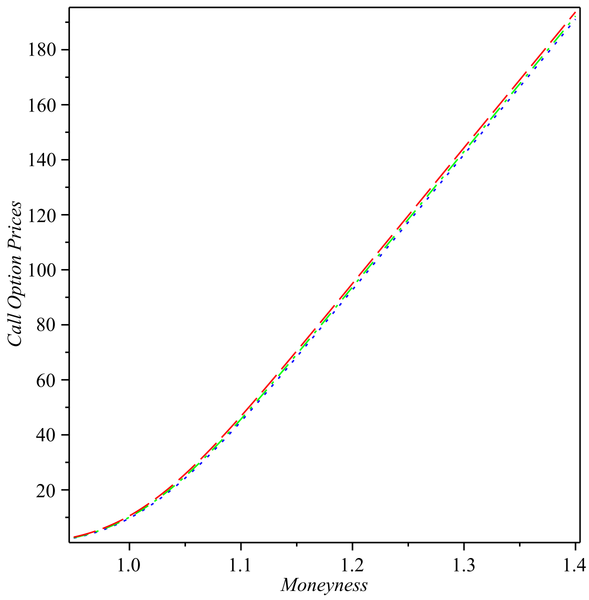


(b)

Figure 6.25: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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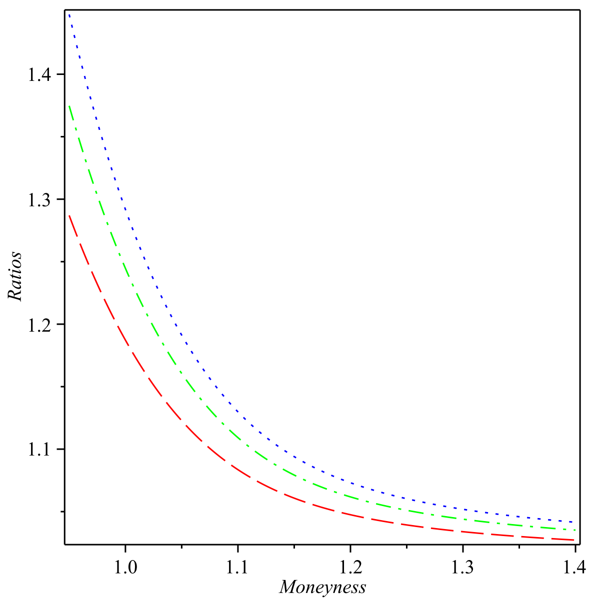
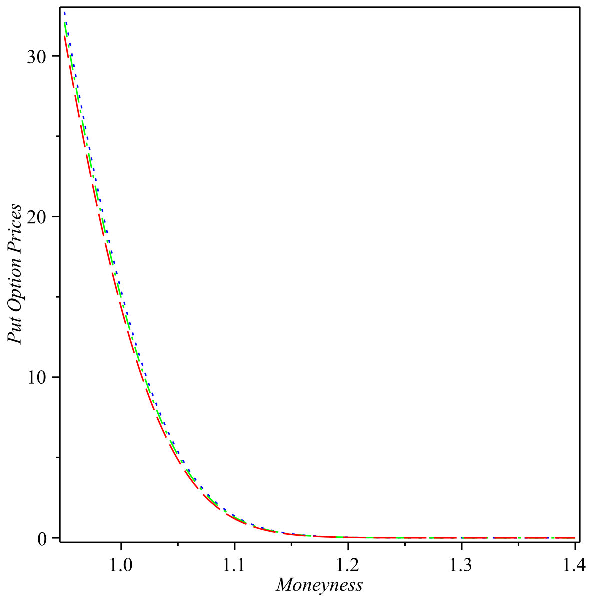


(b)

Figure 6.26: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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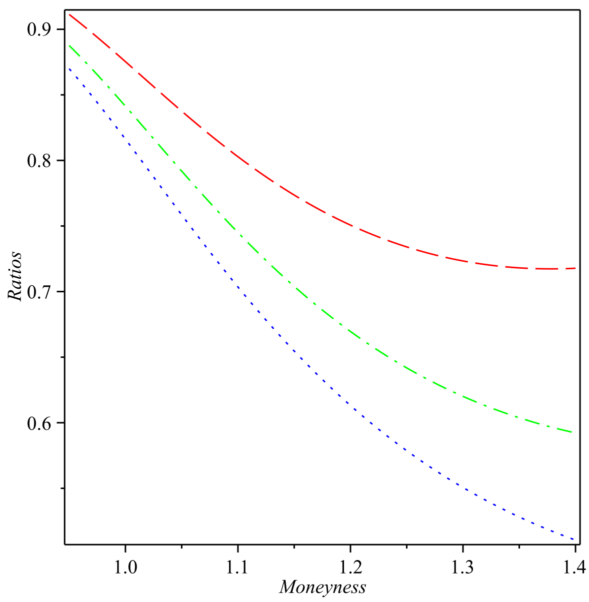
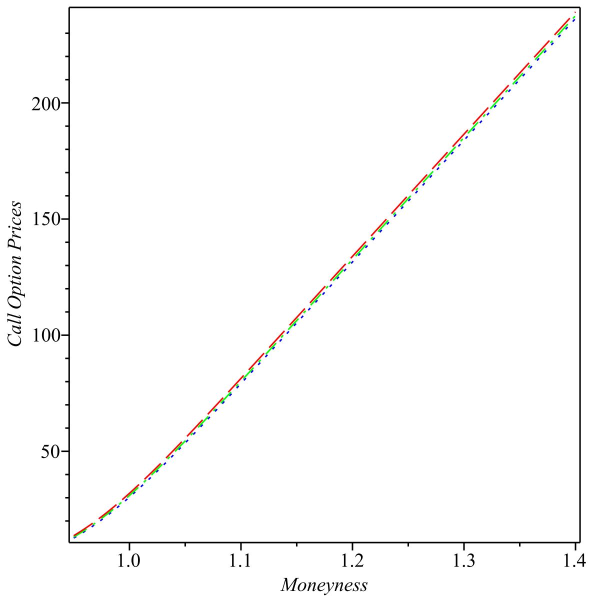
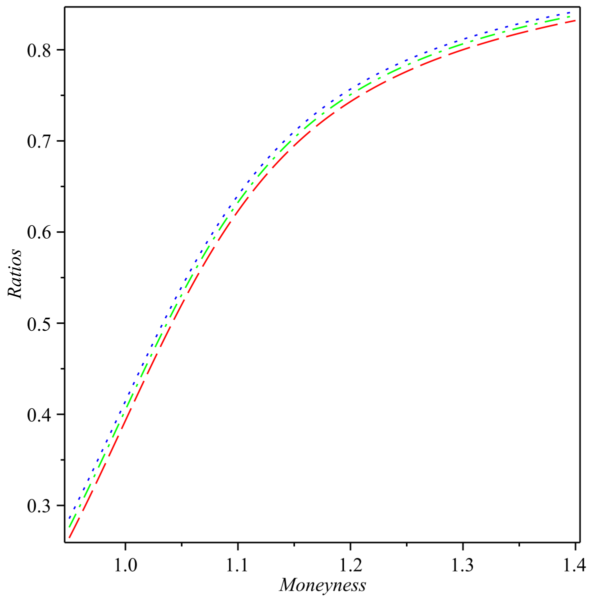


(b)

Figure 6.27: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).

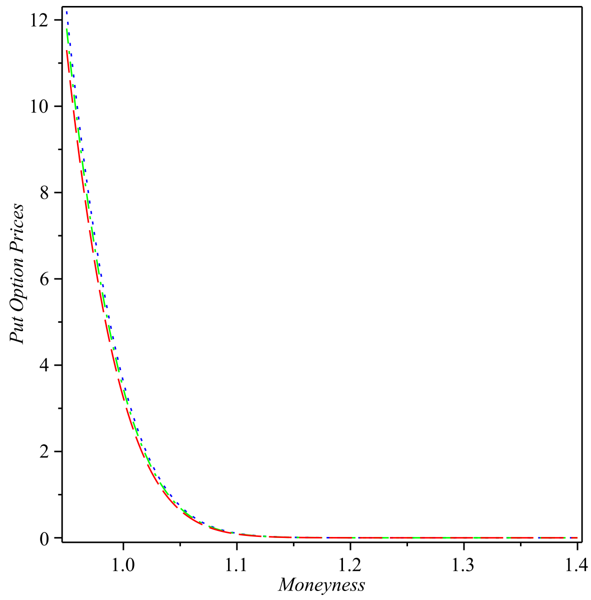


(a)



(b)

Figure 6.28: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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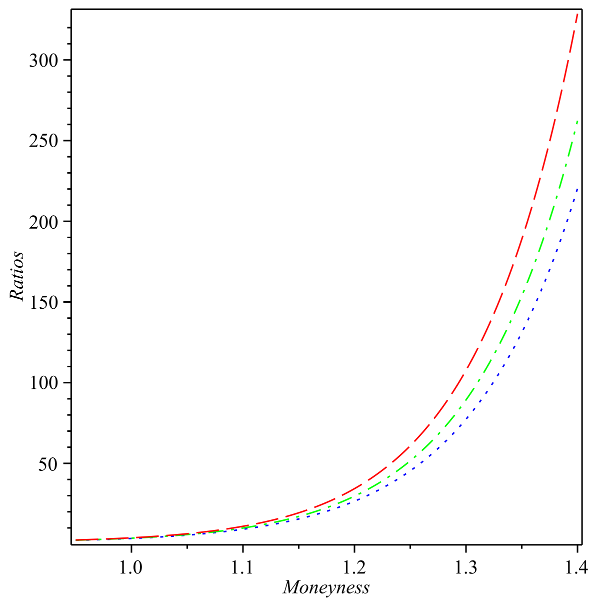
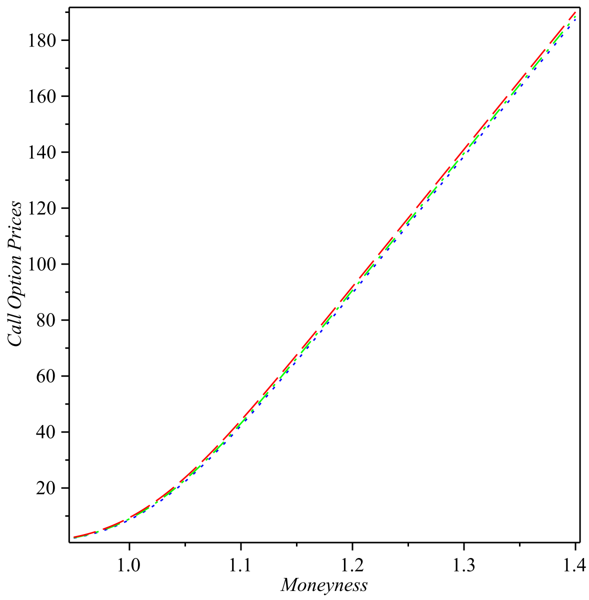
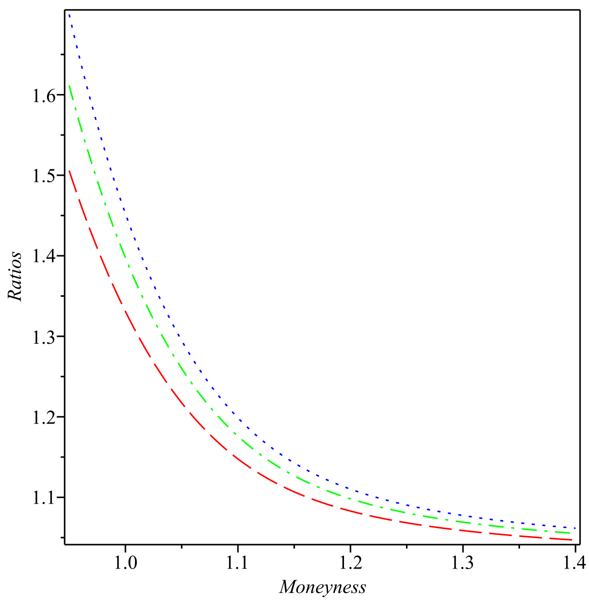


(b)

Figure 6.29: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).

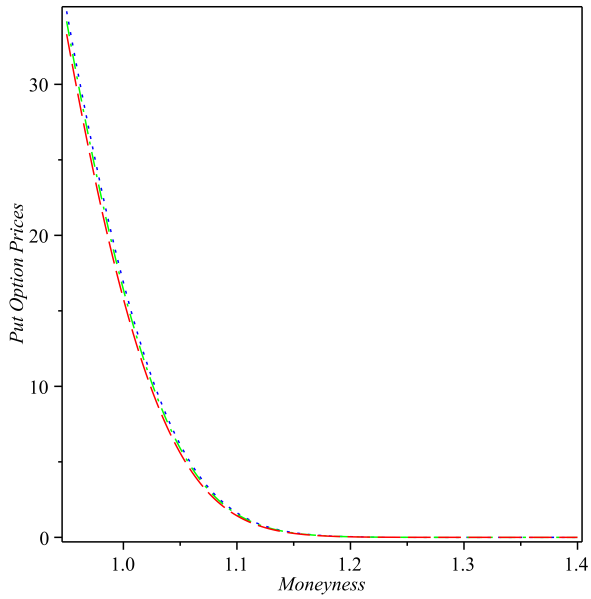


(a)



(b)

Figure 6.30: Sensitivity of call option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).



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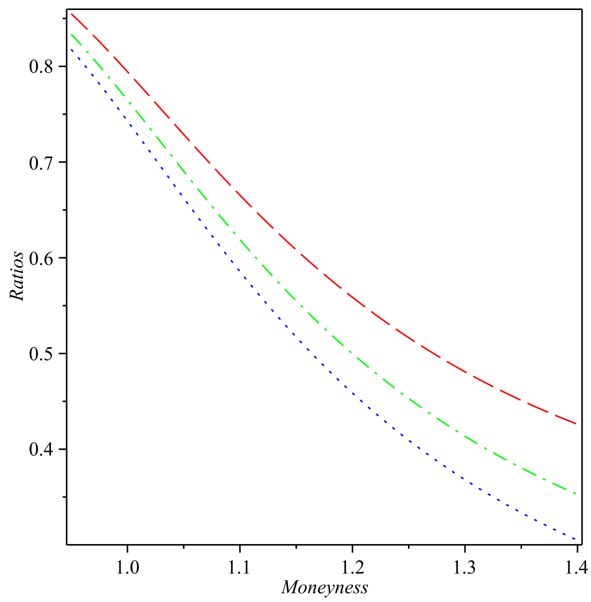


(b)

Figure 6.31: Sensitivity of put option prices to the market price of risk friction coefficient θ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\theta_{\kappa_i} = 0.20$ (dotted line), $\theta_{\kappa_i} = 0.50$ (dash-dotted line), $\theta_{\kappa_i} = 0.90$ (dashed line).

From the results shown in figures 6.14-6.31, the market price of risk friction coefficient slightly affects the option prices. Nevertheless, for some situation, for example in figures 6.20 and 6.21, under the situation of $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$, out-of-the-money options, especially the deep-out-of-the-money options, are impacted by different value of market price of risk friction coefficient. This can interestingly explain the financial market in the real world that market price of risk is the significant variable for the incomplete market for which the friction of the market does exist. However, overall picture of the market price of risk friction coefficient sensitivity test performs slightly effect on option prices. We will ignore this parameter for the next parameter test of sensitivity.

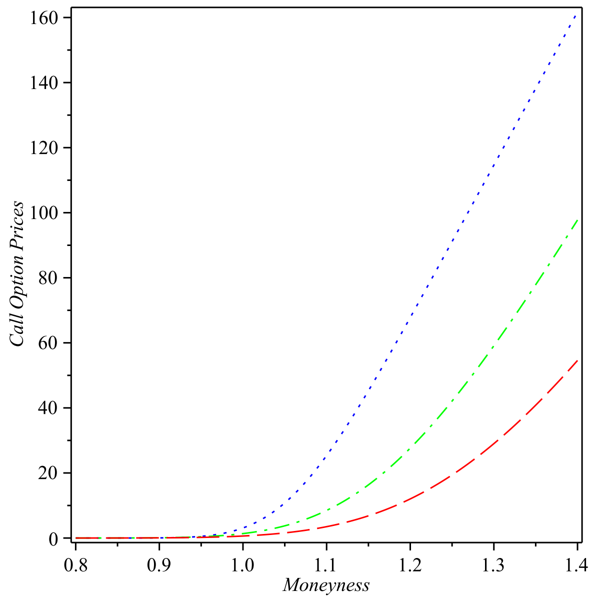
6.2.4 Maturity (T)

Maturity or maturity date is the last due date for the payment of a financial contract or instrument (in our thesis refers to an option contract), in other words, maturity is the end of the life of a contract of the option [2]. Definitely, the maturity is the parameter to be considered in the pricing model as it affects the decision of the investors to manage their risks.

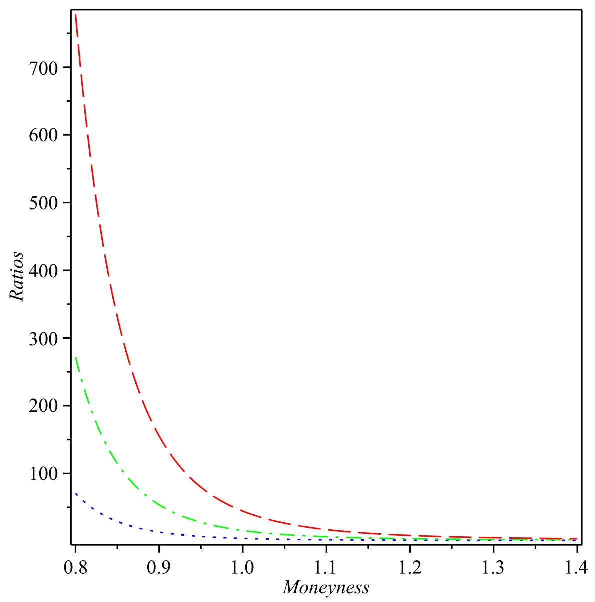
Table 6.6: Parameter values used in the sensitivity analysis of the model with respect to the maturity

$\sigma_S = 0.20$	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\theta_{\kappa 1} = 0.45$
	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 3} = 0.05$	$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$

In this section, we investigate the sensitivity of the model output to the change of maturity by setting the parameters at the values as shown in the table 6.6. By changing the parameter maturity T at three different values: $T = 0.1$, $T = 0.3$ and $T = 0.5$, we observe the sensitivity of option prices to maturity date. The situation of the market is set to be the same as that in previous section since the friction coefficient of market price of risk has slightly impact on the sensitivity. The simulation results are shown in Figure 6.32-6.49.

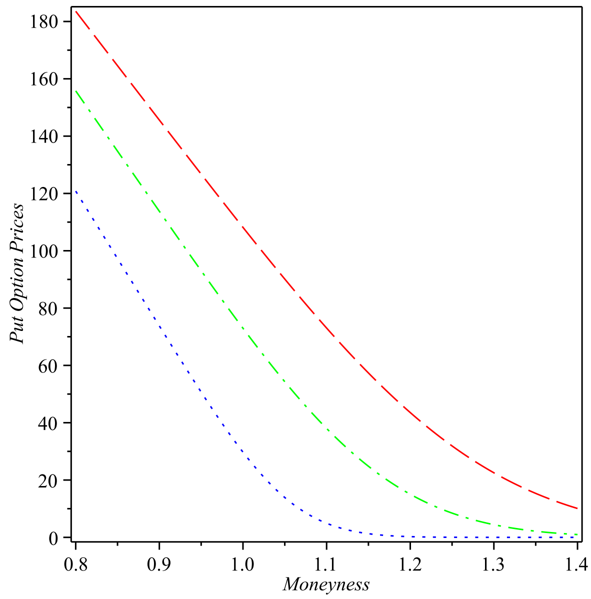


(a)



(b)

Figure 6.32: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).



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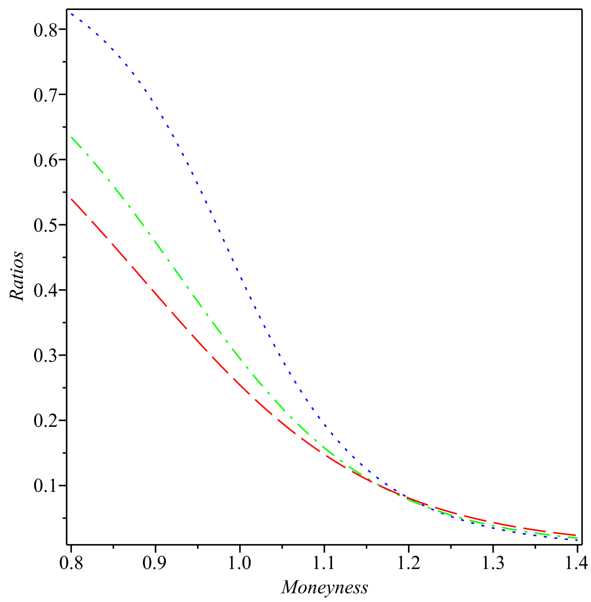
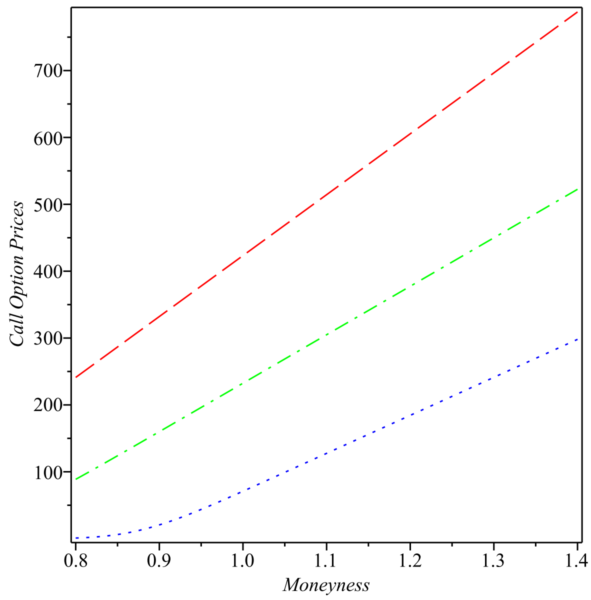


(b)

Figure 6.33: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).



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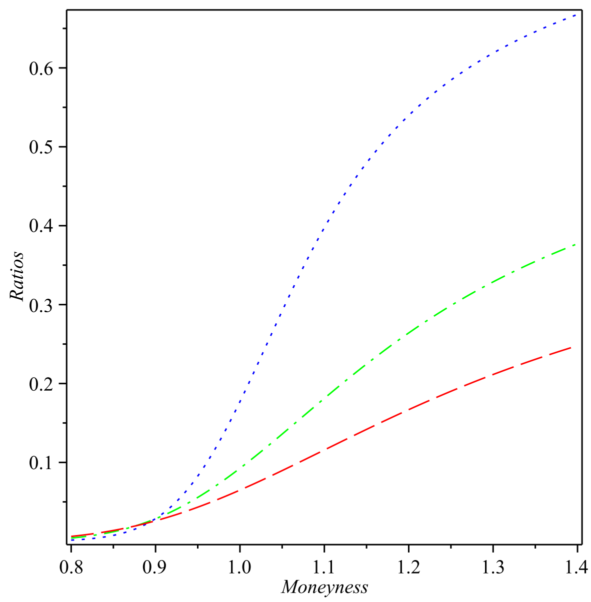
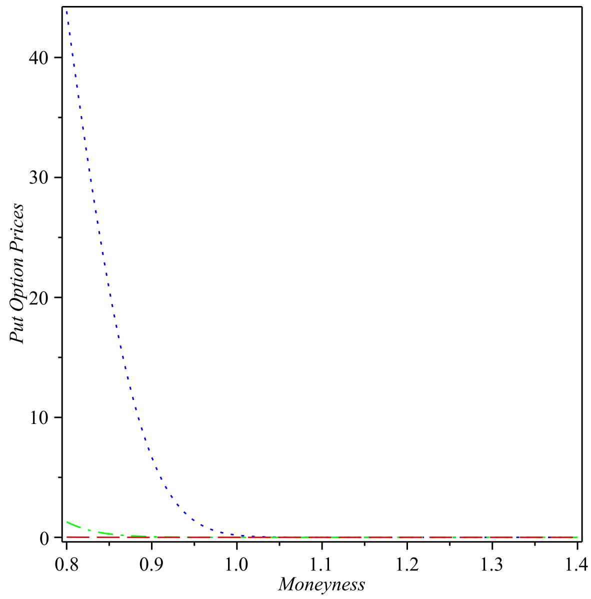


(b)

Figure 6.34: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).



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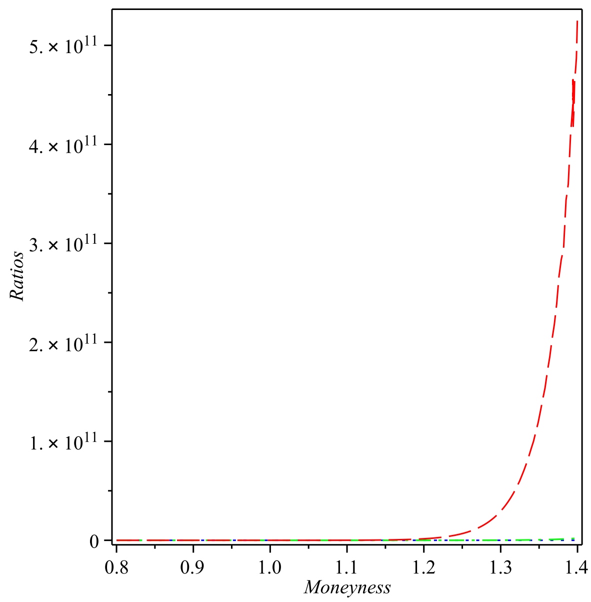
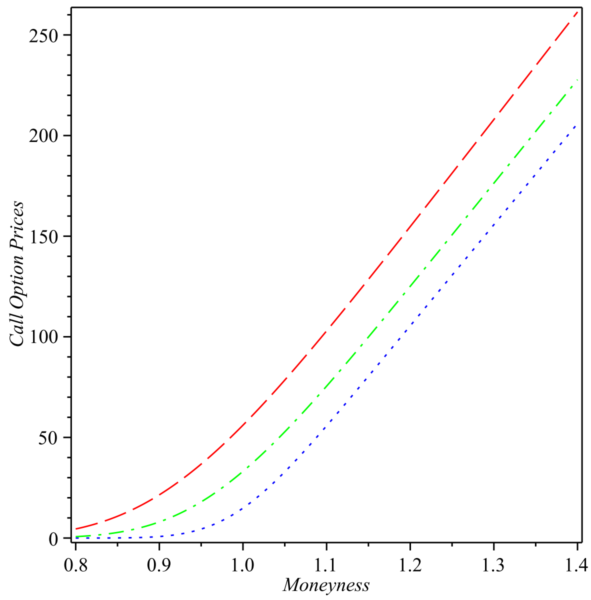
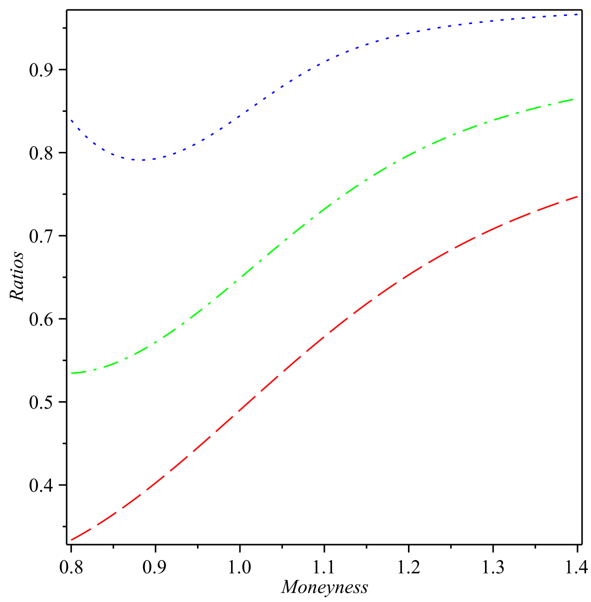


(b)

Figure 6.35: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

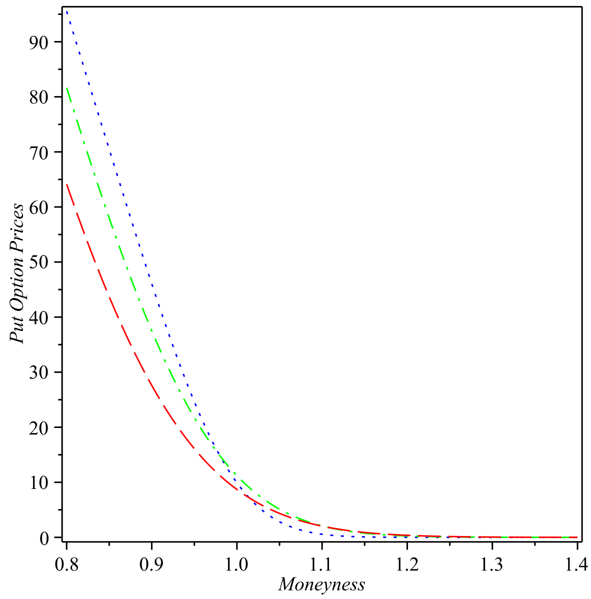


(a)

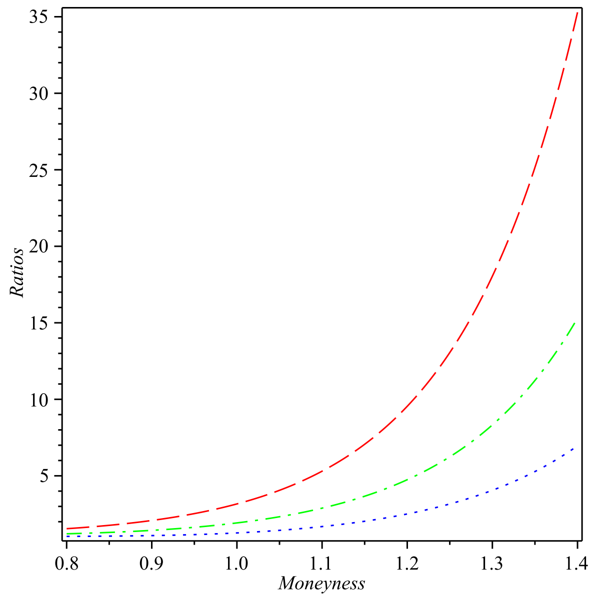


(b)

Figure 6.36: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

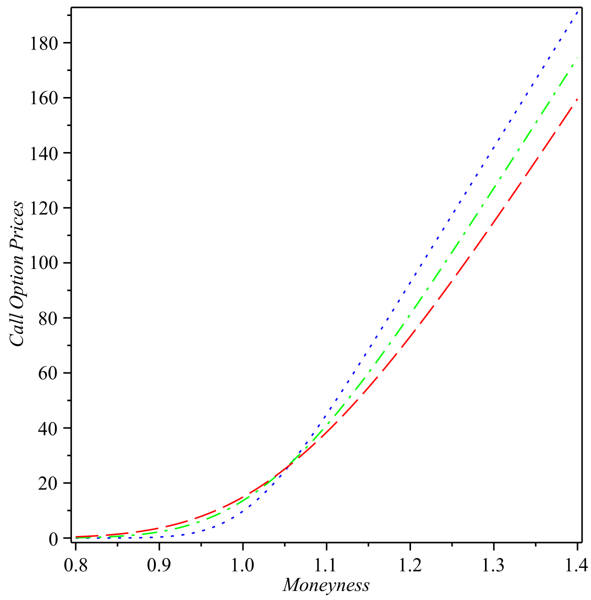


(a)

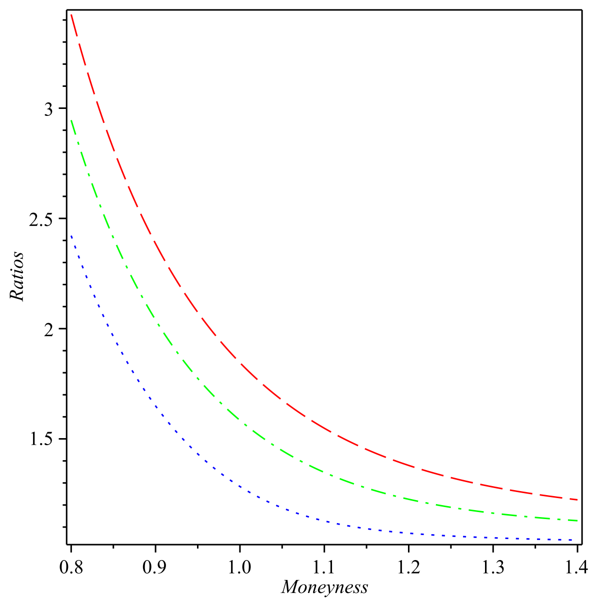


(b)

Figure 6.37: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

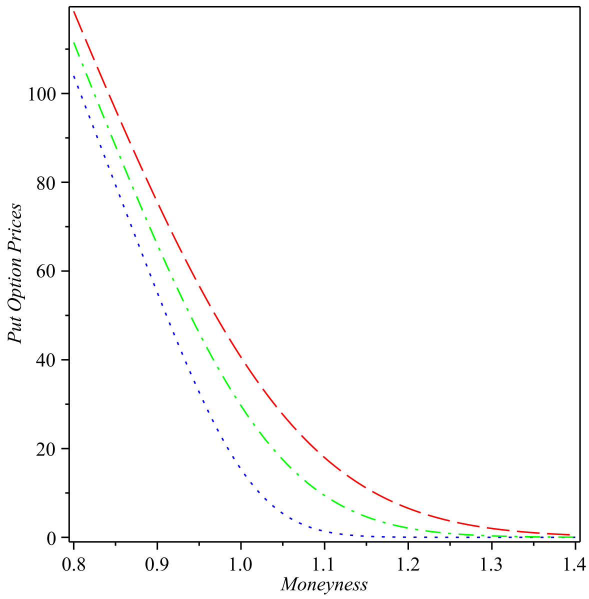


(a)

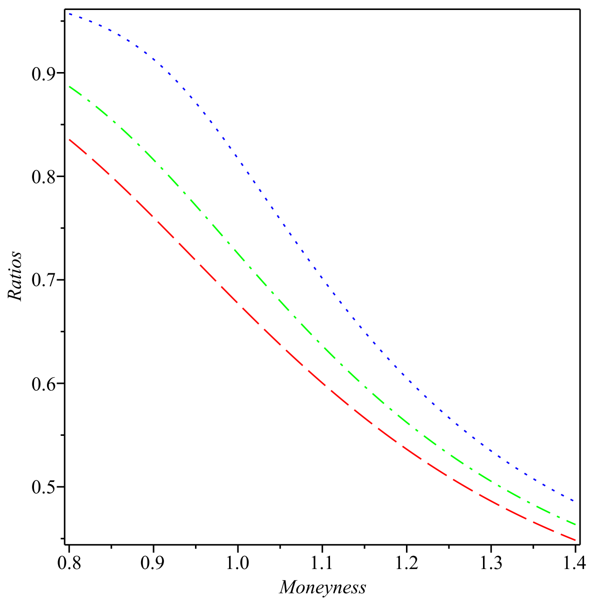


(b)

Figure 6.38: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

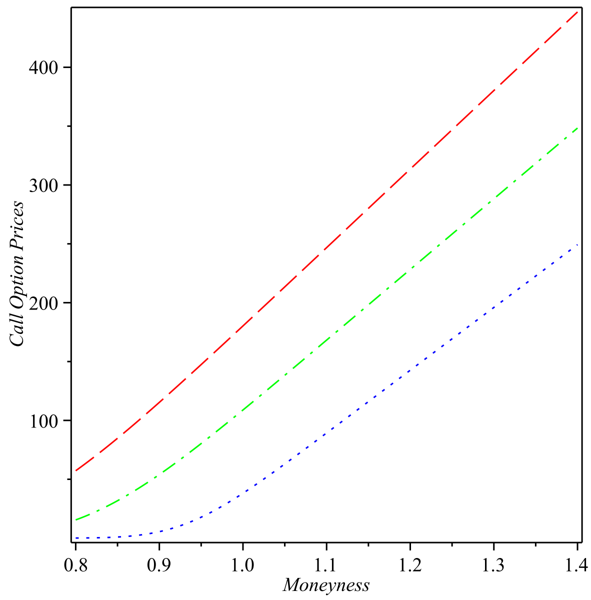


(a)

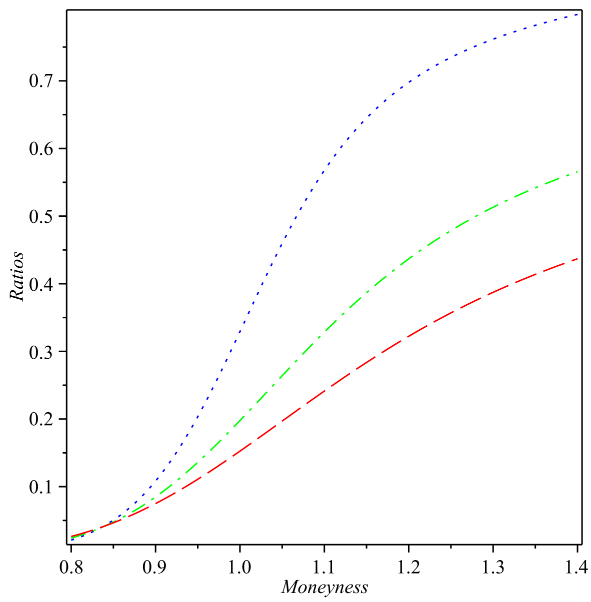


(b)

Figure 6.39: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

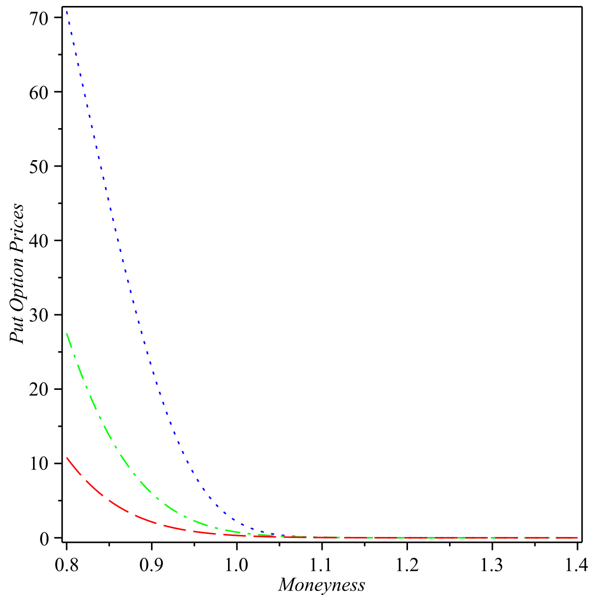


(a)



(b)

Figure 6.40: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).



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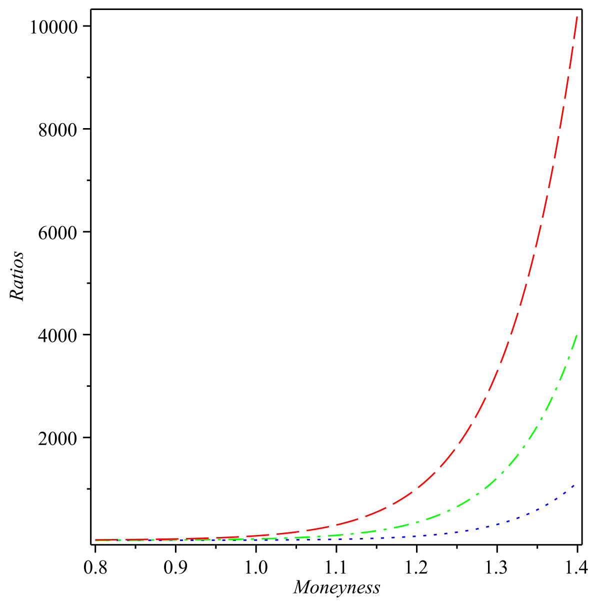
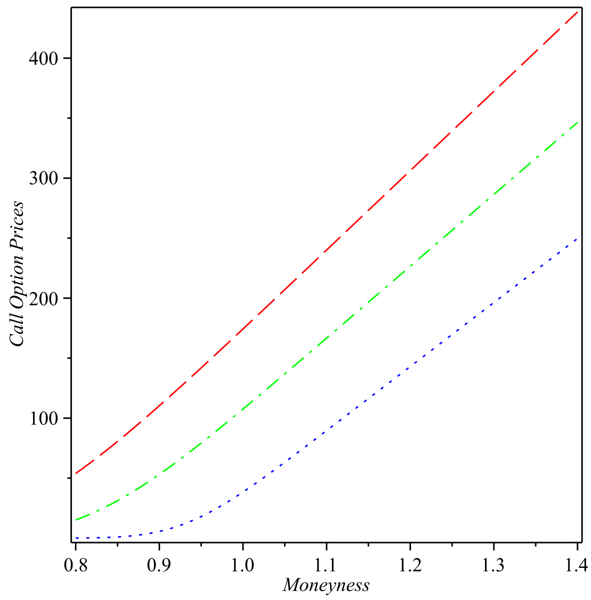
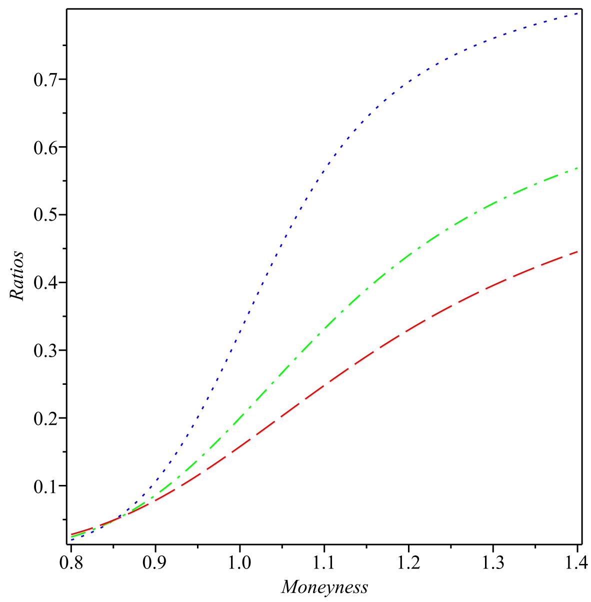


(b)

Figure 6.41: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

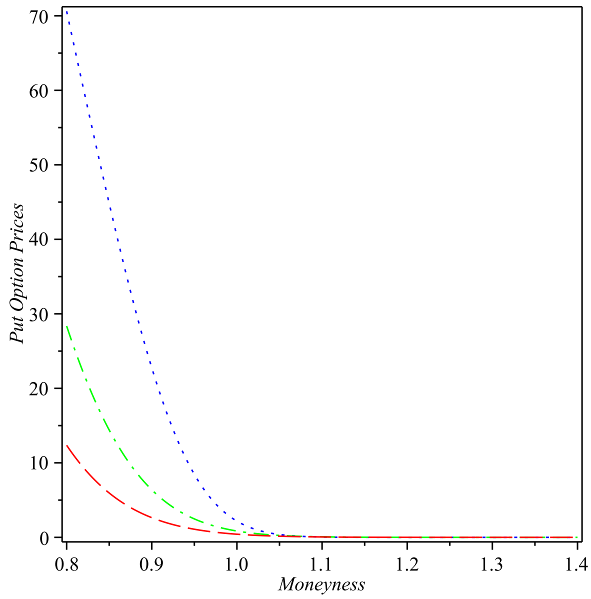


(a)

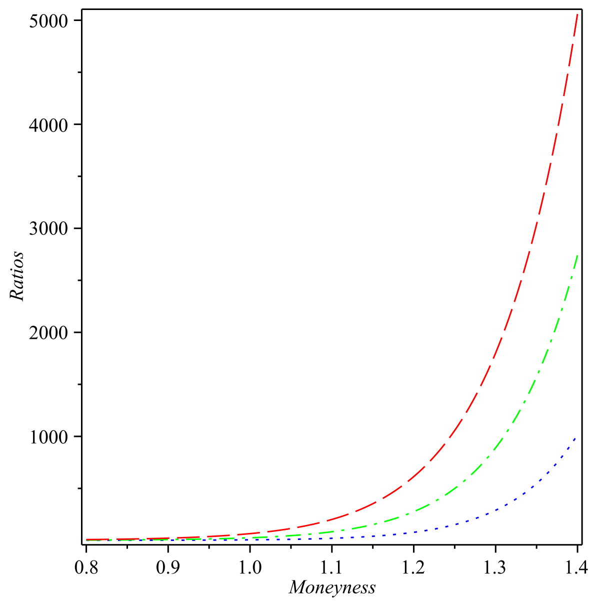


(b)

Figure 6.42: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

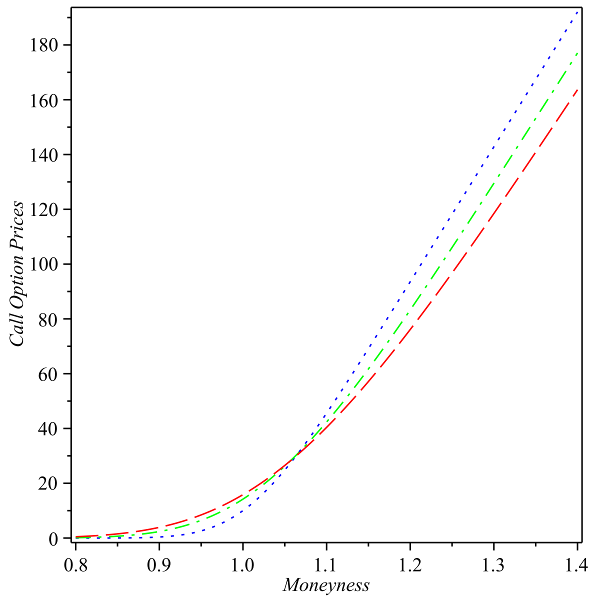


(a)

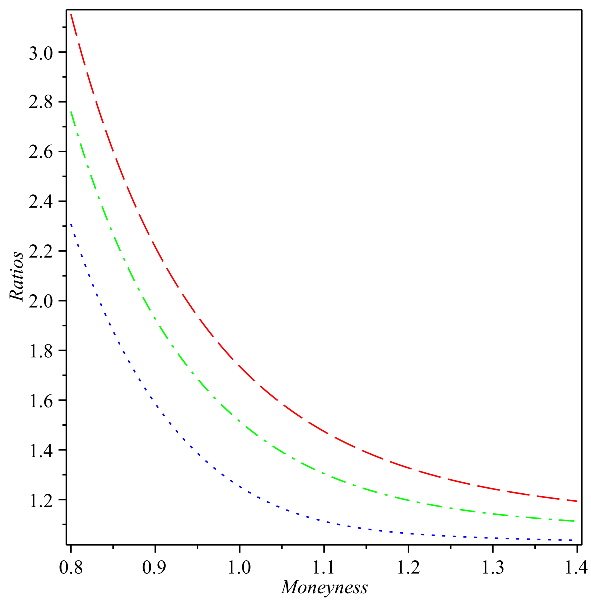


(b)

Figure 6.43: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

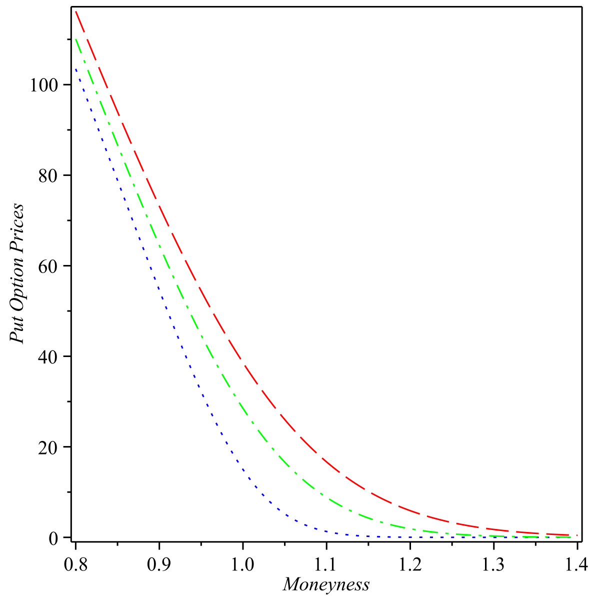


(a)

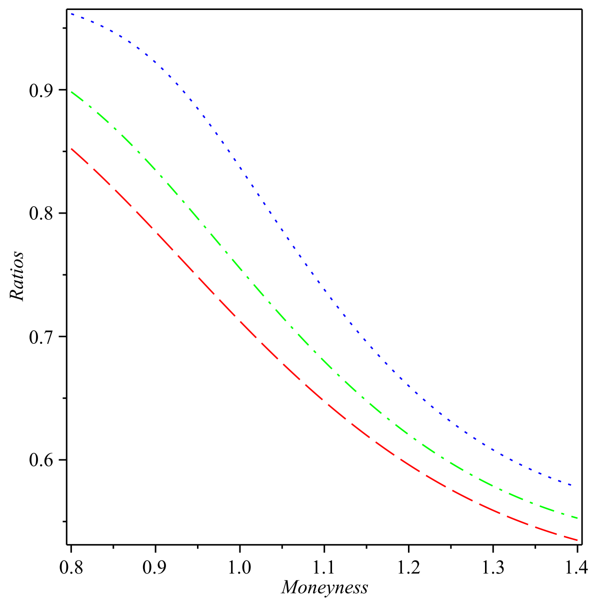


(b)

Figure 6.44: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

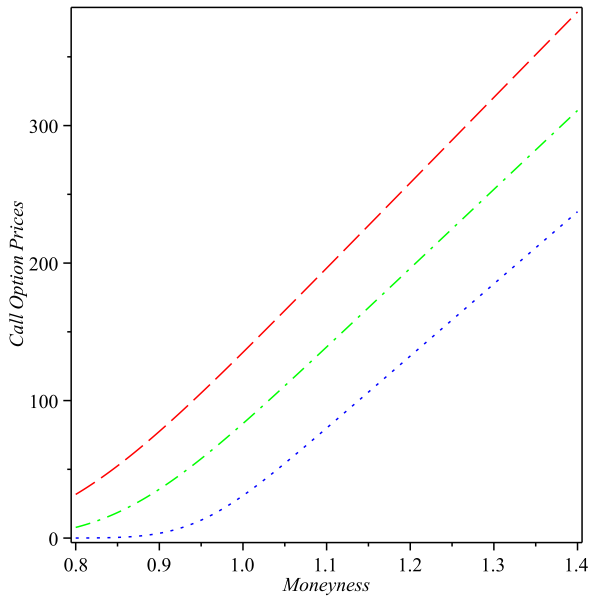


(a)

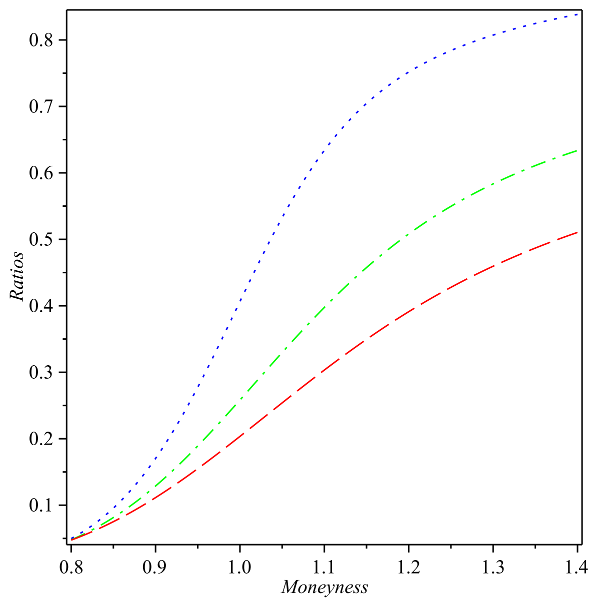


(b)

Figure 6.45: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

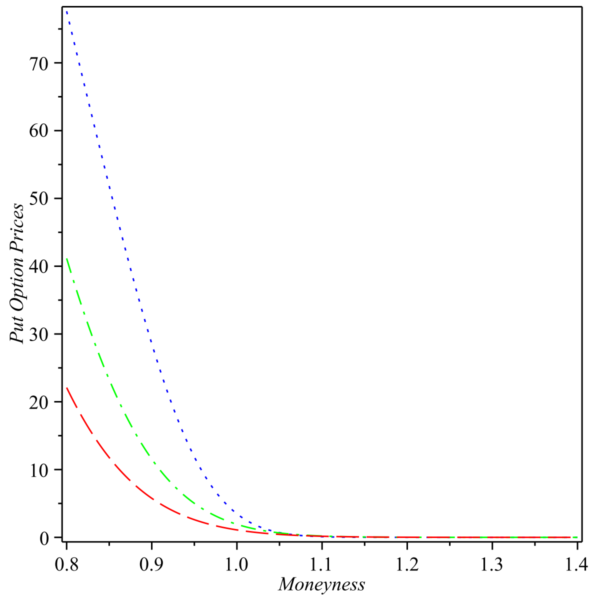


(a)

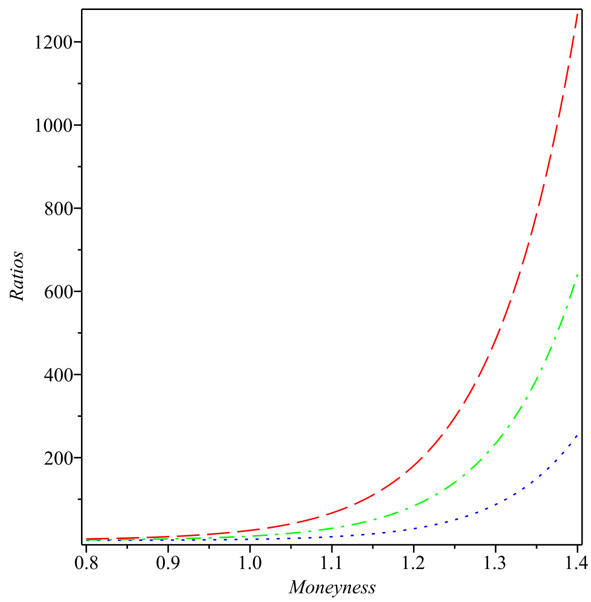


(b)

Figure 6.46: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

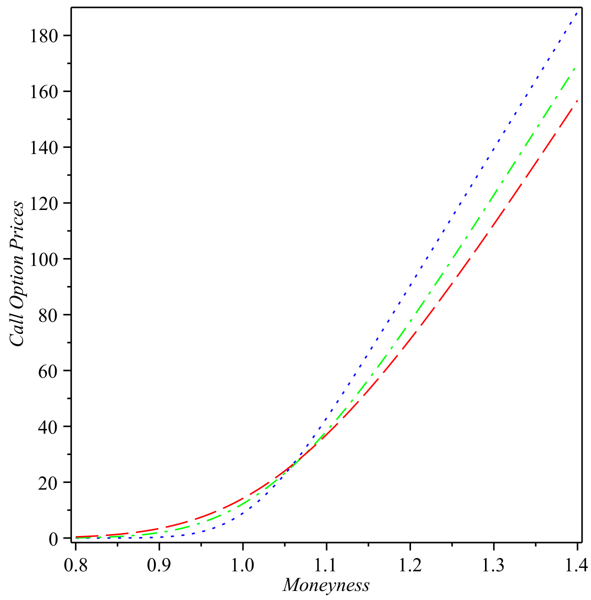


(a)

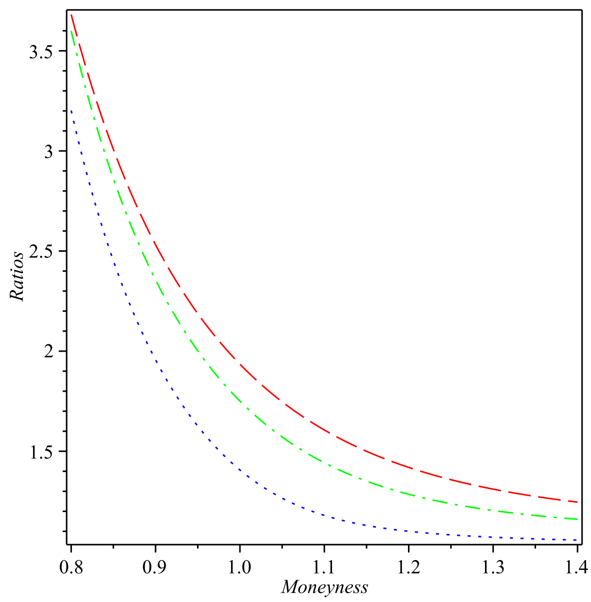


(b)

Figure 6.47: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

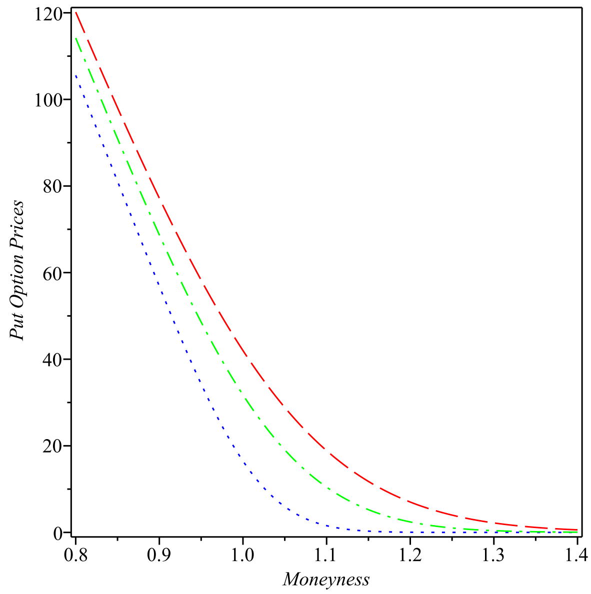


(a)

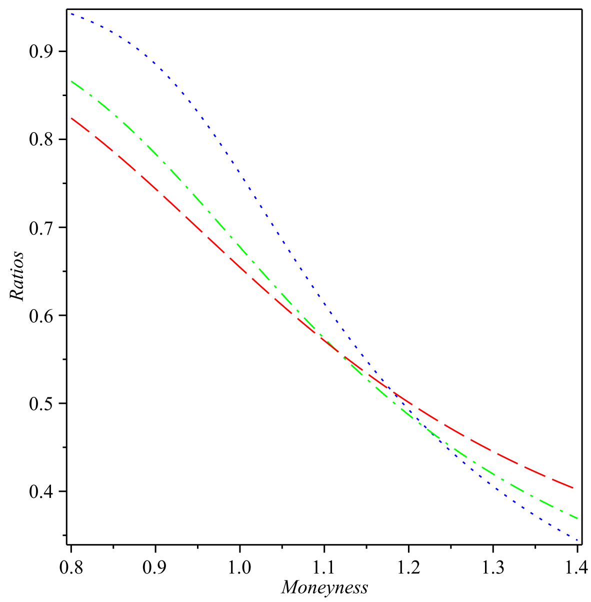


(b)

Figure 6.48: Sensitivity of call option prices to the maturity T for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).



(a)



(b)

Figure 6.49: Sensitivity of put option prices to the maturity T for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $T = 0.1$ (dotted line), $T = 0.3$ (dash-dotted line), $T = 0.5$ (dashed line).

The results shown in the figures include two categories: Option Prices, and the ratio of the Option Price determined by moneyness and the option price obtained by our model.

From the first category of results, it is clear that by changing the value of maturity, the option prices change significantly. However, by the uncertainty property of the model, there is no exact trend to be performed. As the maturity time increases, the option prices may decreases or increases as we expect from the stochastic property. In figure 6.32-6.33, under the situation of $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$, as the maturity increases, the call option prices decrease and the put option prices increase. In figures 6.34-6.35 under the situation of $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$, as the maturity time increases, the call option prices increase and the put option prices decrease. Moreover, when the maturity time increases, the option prices may increase or decrease based upon the moneyness. For example, in figure 6.37 when $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$, in-the-money put option prices decrease when the maturity time increases. The results are contrast, under the out-of-the-money put option. Similar situation occurs in call option prices as shown in Figure 6.38.

From the second type of figures, under the stochastic property, call options may be either overvalued or undervalued depending on the situation of the financial market or the market friction in each source. Under the friction of $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$, in figure 6.32 and 6.33, the call option prices are always overvalued, whereas the put option prices are always undervalued. Conversely, if the situation changes to $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$, in figures 6.40 and 6.41, the call option prices are always undervalued, whereas the put option prices are always overvalued. The reason for these results is that the stochastic dividend yield, stochastic earning yield and stochastic market price of risk all affect the present value of the future dividends.

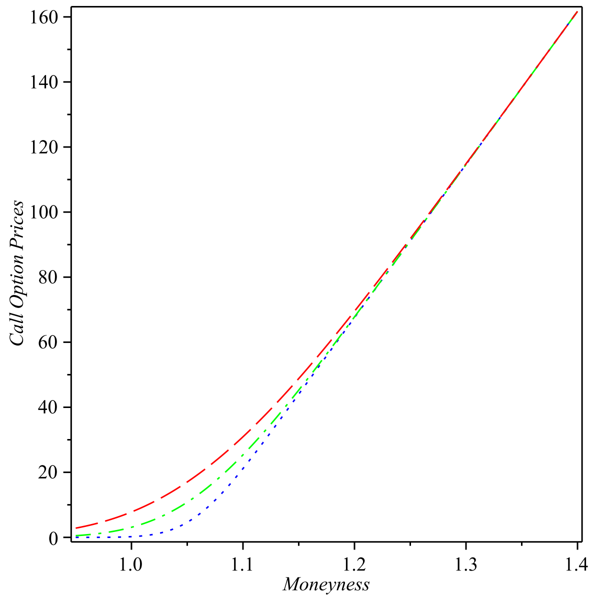
Under the market situation of $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$, in figures 6.32 and 6.33, for the in-the-money and deep-in-the-money call options, different level of maturity has almost no impact on the prices; whereas, the in-the-money and deep-in-the-money put options are impacted. Conversely, when $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$, in figures 6.40 and 6.41, the maturity time has a significant impact on the deep-in-the-money call options and a huge impact to the deep-out-of-the-money put options. These results are owing to the stochastic character of the model.

6.2.5 Stock Volatility (σ_S)

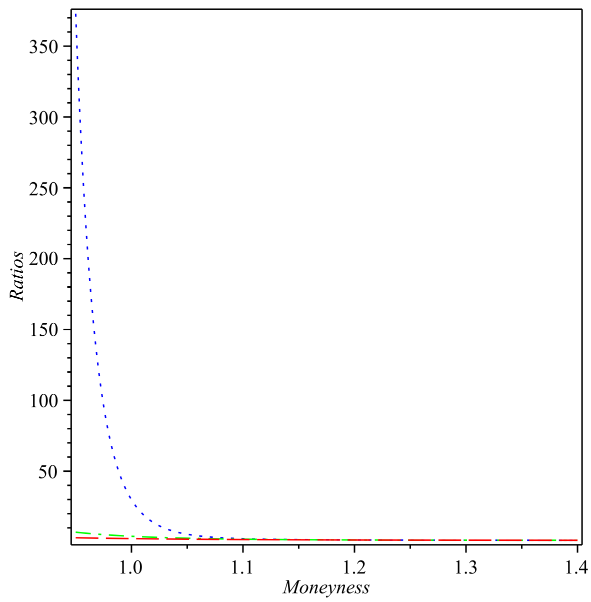
In this section, we will test the results of the simulations for sensitivity of option prices to the underlying volatility. The same method is applied for this testing by using different market frictions. We set the parameters at the values as shown in the table 6.7. By changing the value of the stock volatility at three levels: $\sigma_S(SV) = 0.1$, $\sigma_S(SV) = 0.2$ and $\sigma_S(SV) = 0.3$, the simulation results are obtained for the analysis. The simulation results are shown in figures 6.50-6.67.

Table 6.7: Parameter values used in the sensitivity analysis of the model with respect to the stock volatility

$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\theta_{\kappa 1} = 0.45$
	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 3} = 0.05$	$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 1} = 0.10$

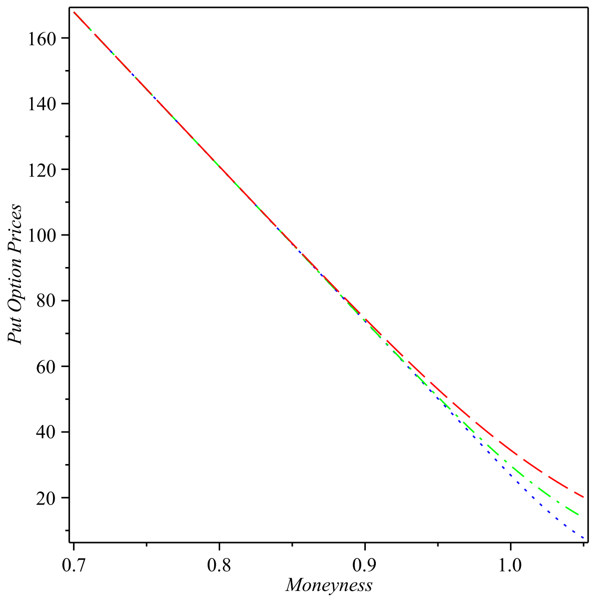


(a)

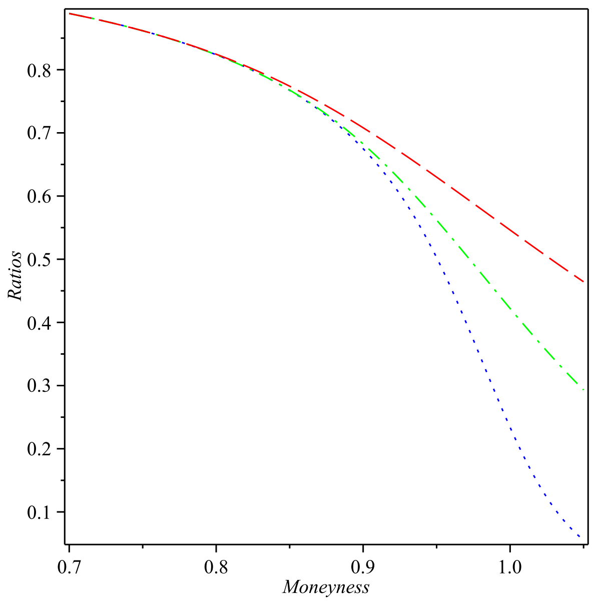


(b)

Figure 6.50: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

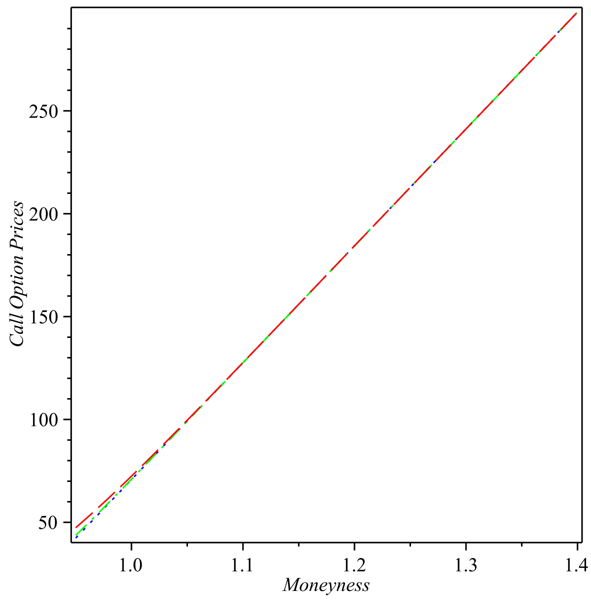


(a)



(b)

Figure 6.51: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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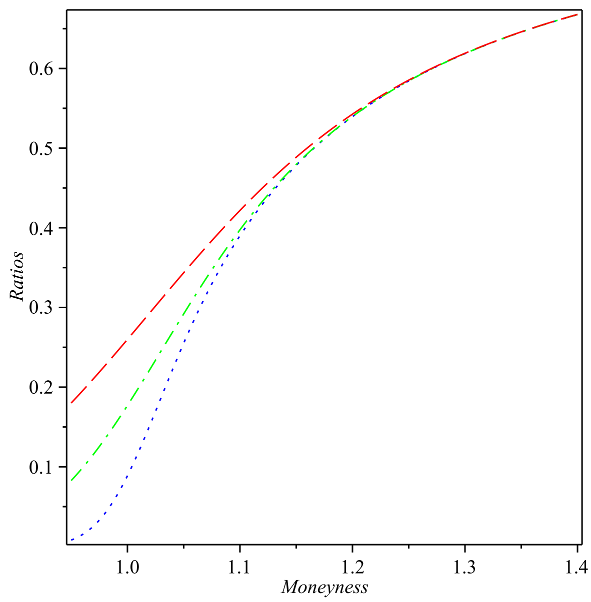
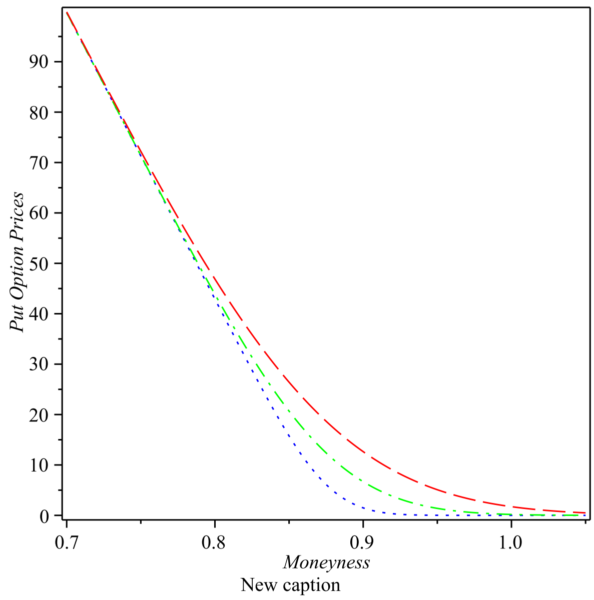
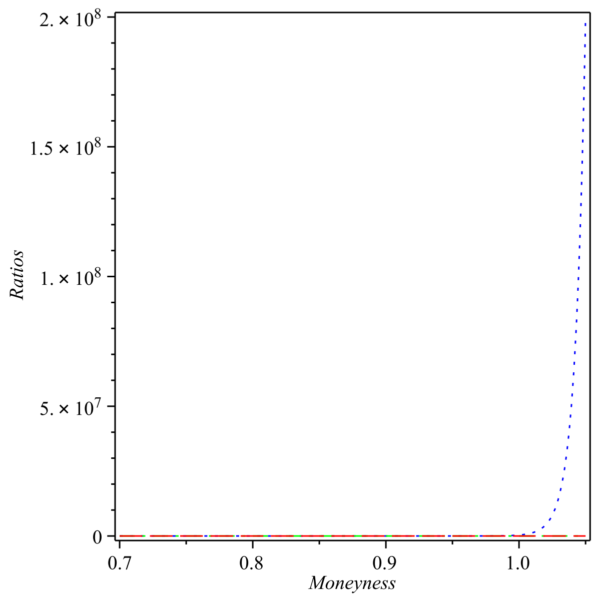


(b)

Figure 6.52: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

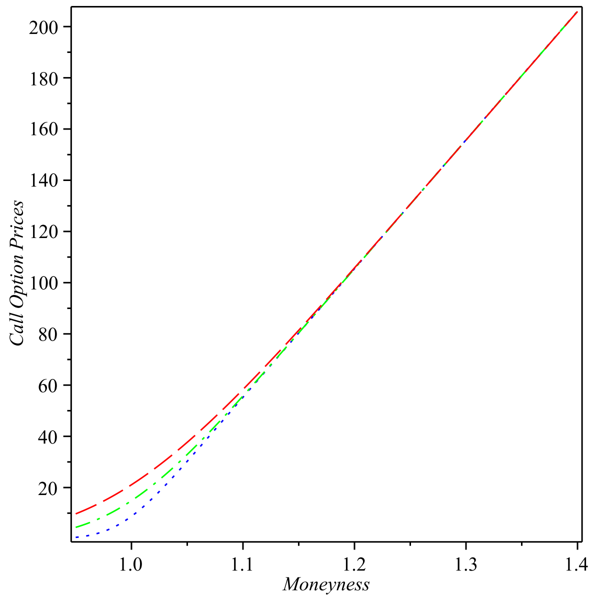


(a)



(b)

Figure 6.53: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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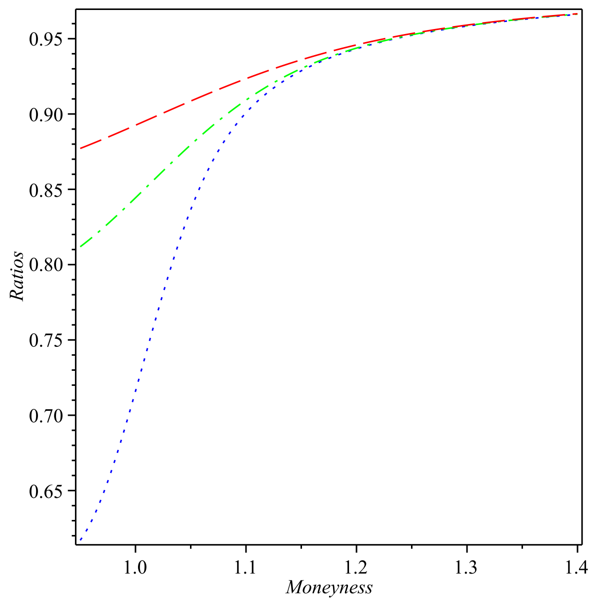
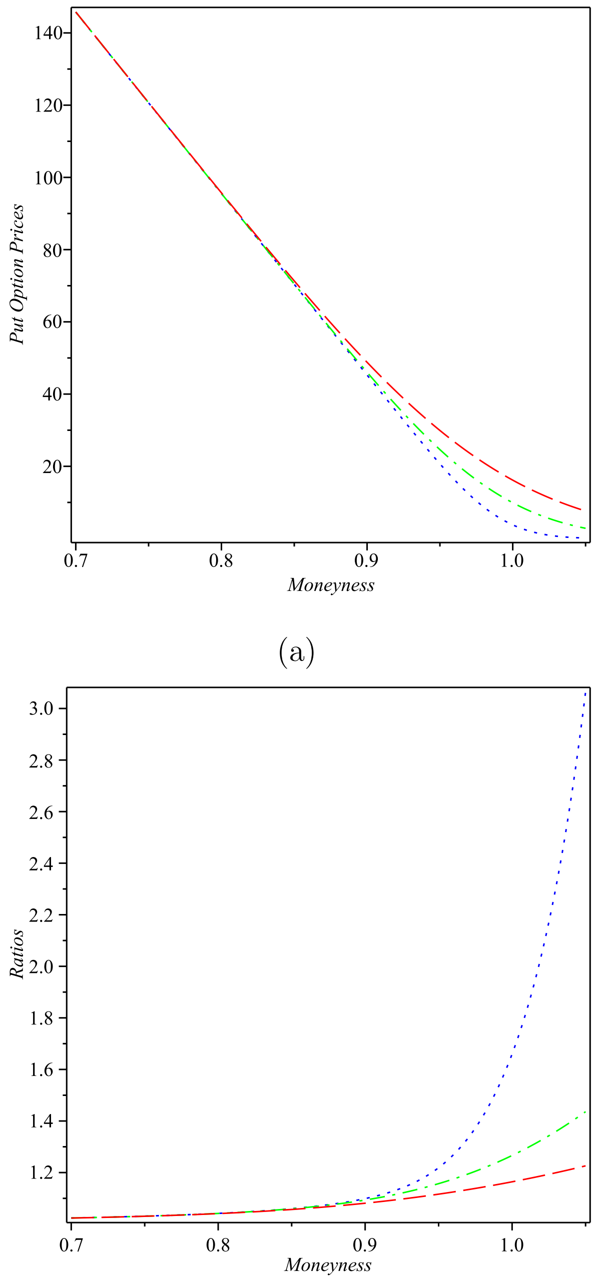


(b)

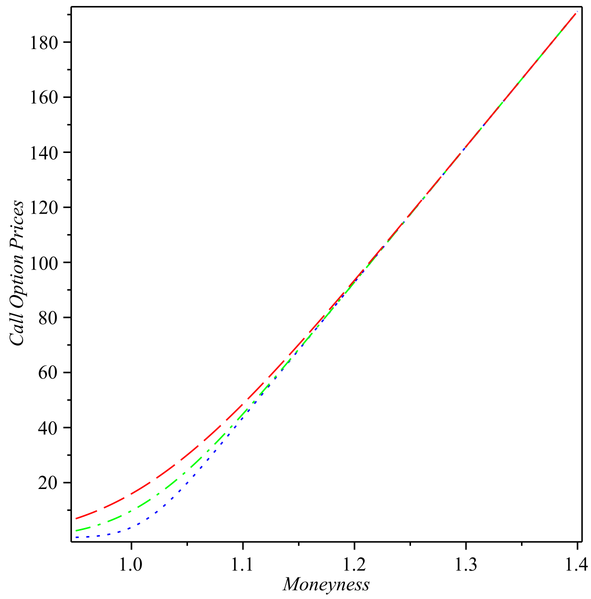
Figure 6.54: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



(a)

(b)

Figure 6.55: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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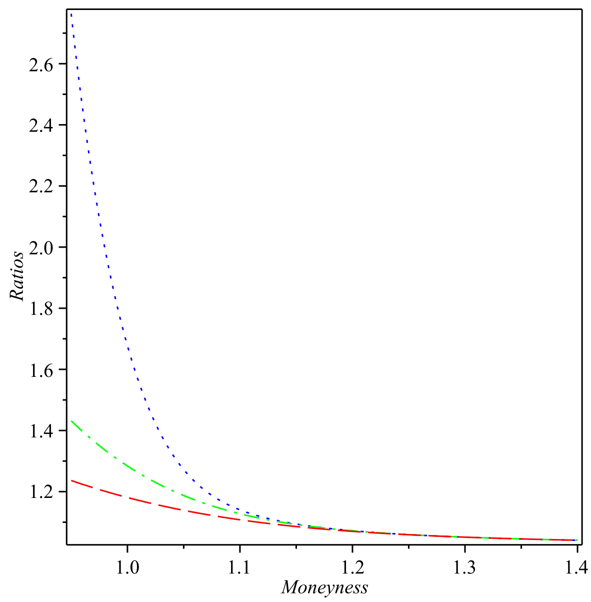
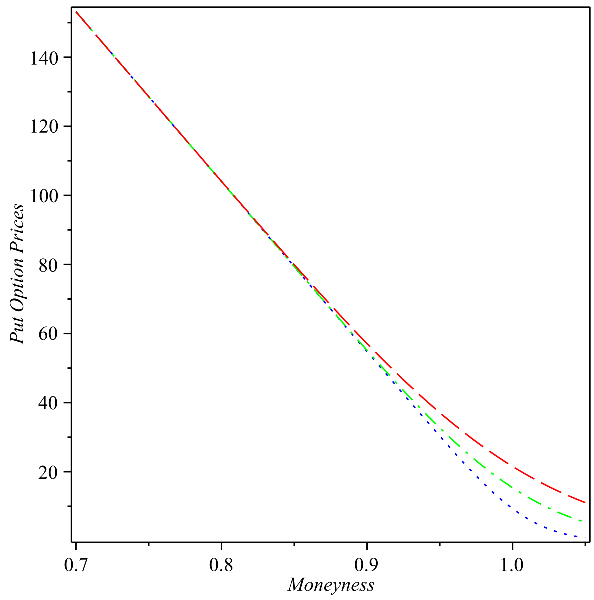


(b)

Figure 6.56: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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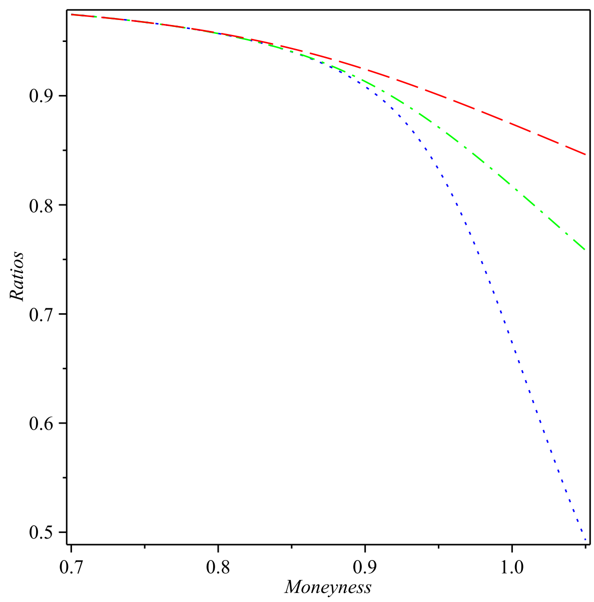
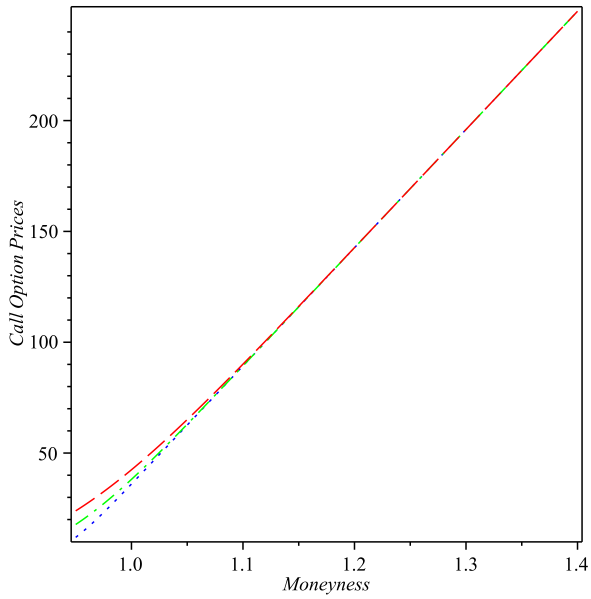
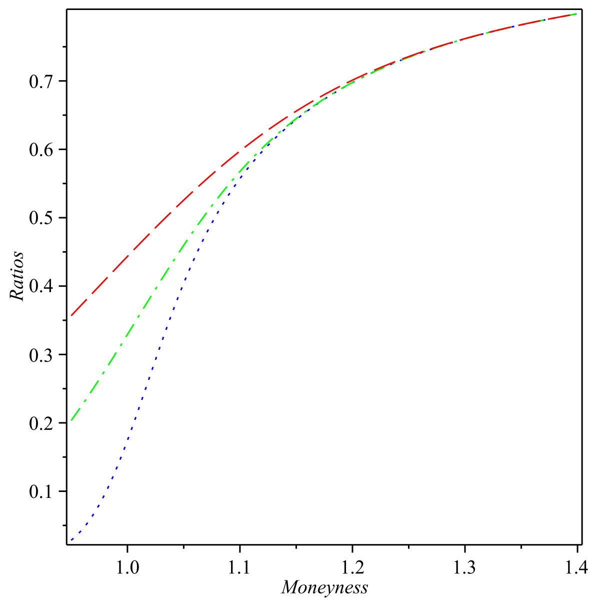


(b)

Figure 6.57: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

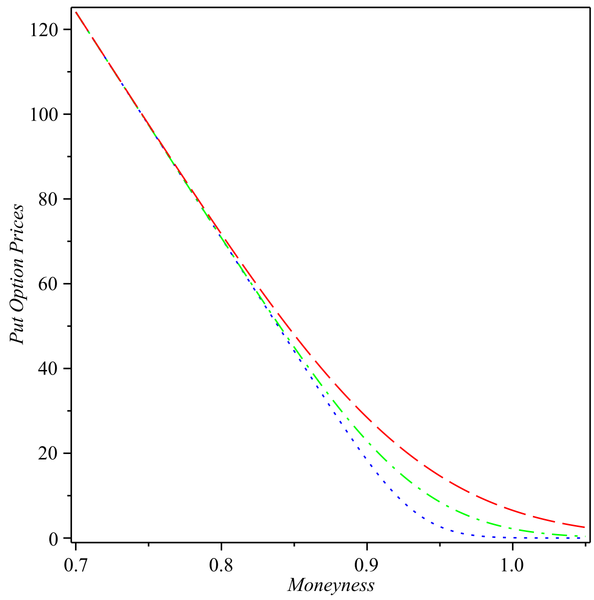


(a)

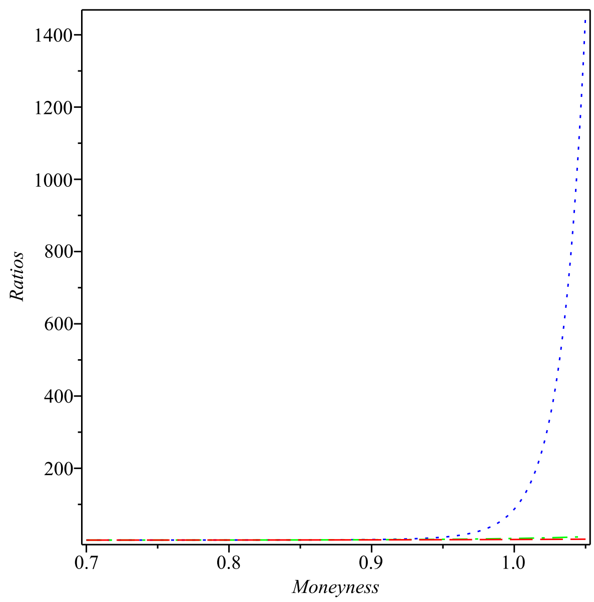


(b)

Figure 6.58: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

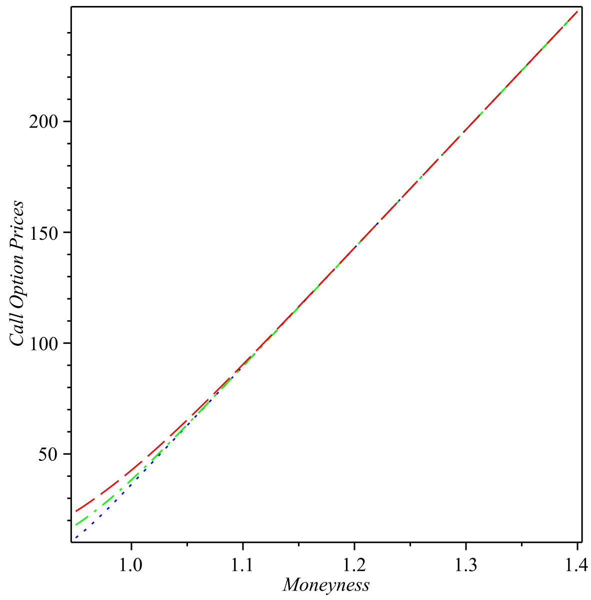


(a)



(b)

Figure 6.59: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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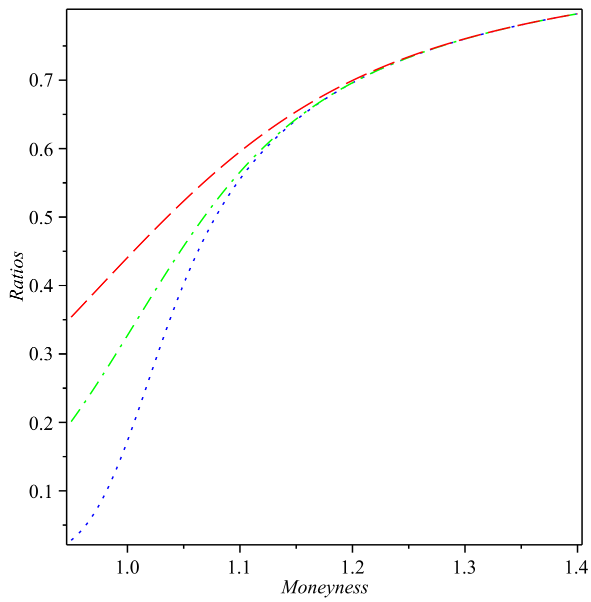
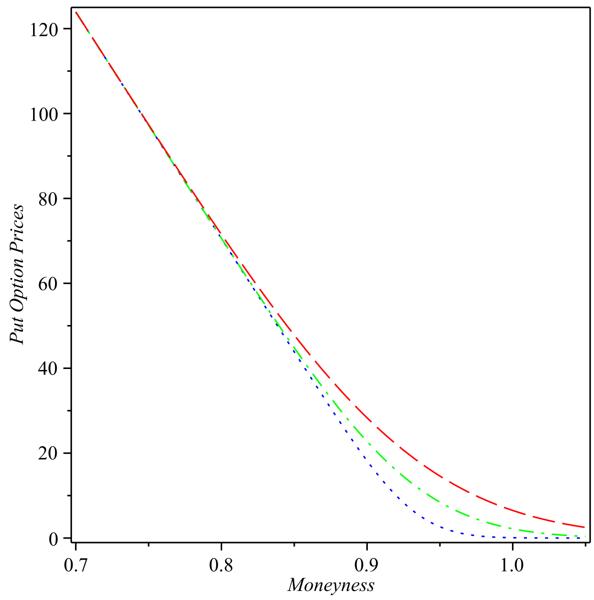


(b)

Figure 6.60: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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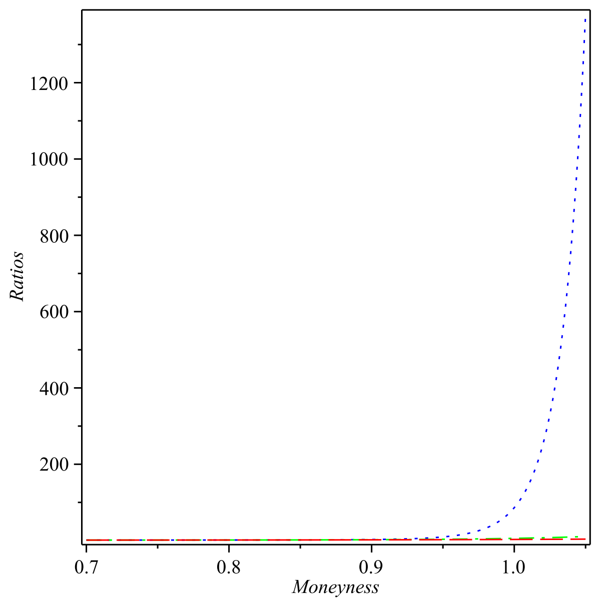
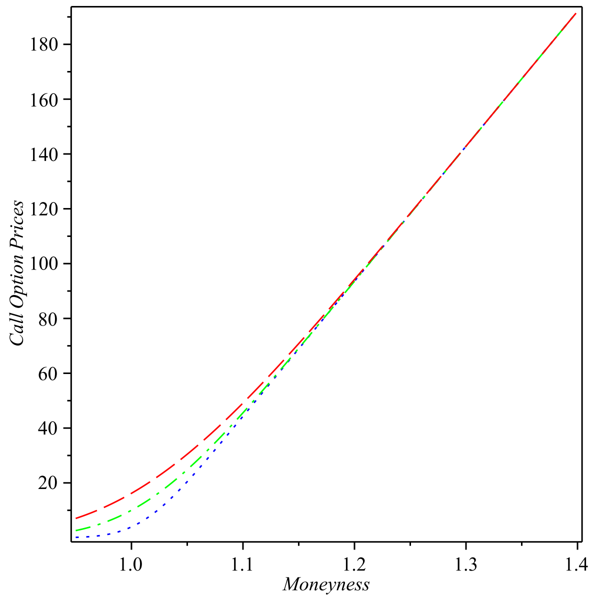


(b)

Figure 6.61: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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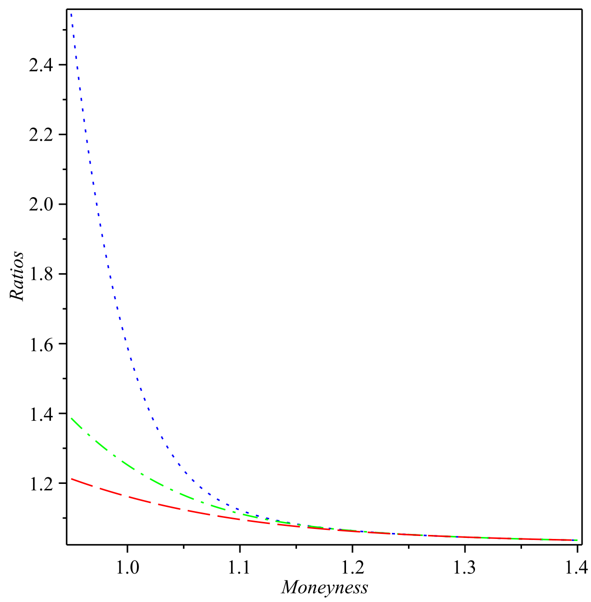
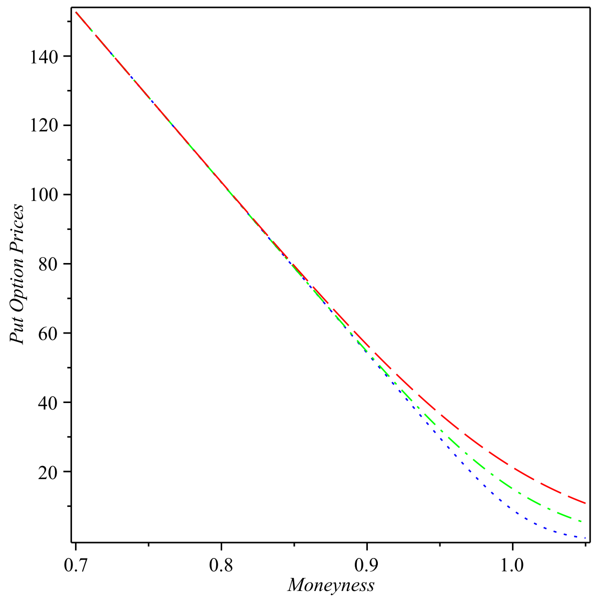


(b)

Figure 6.62: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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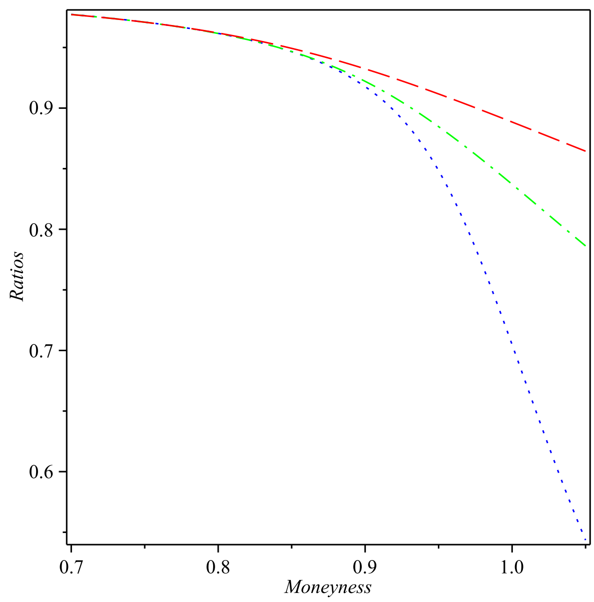
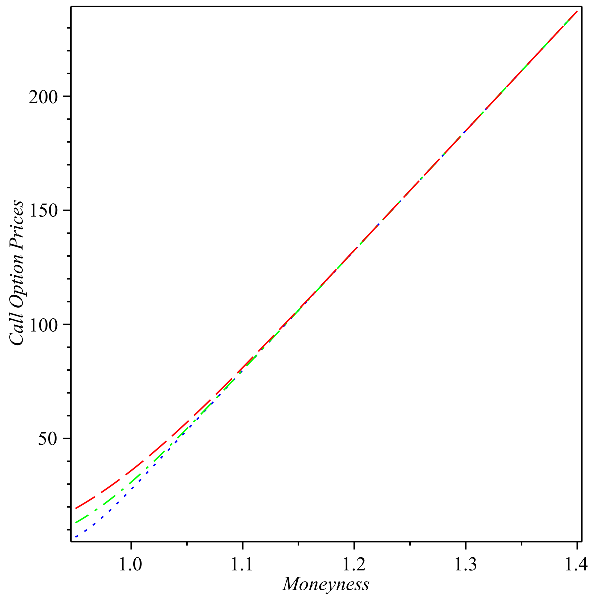


(b)

Figure 6.63: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



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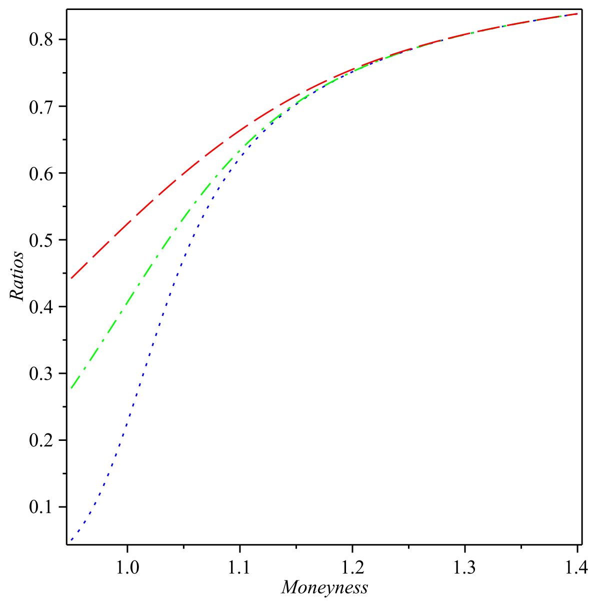
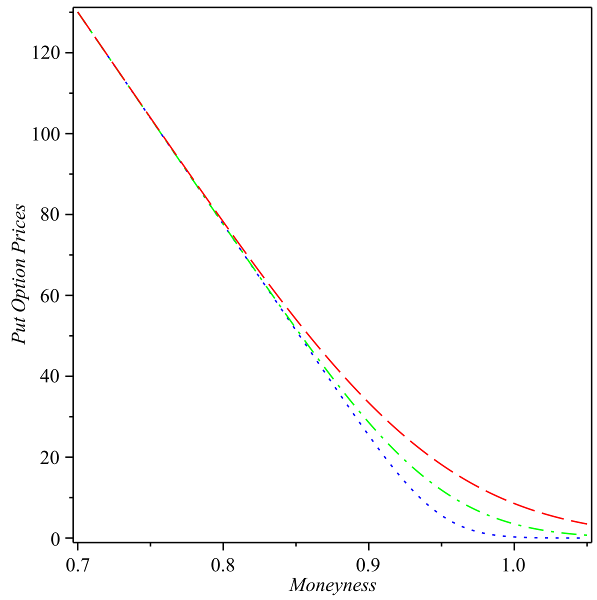
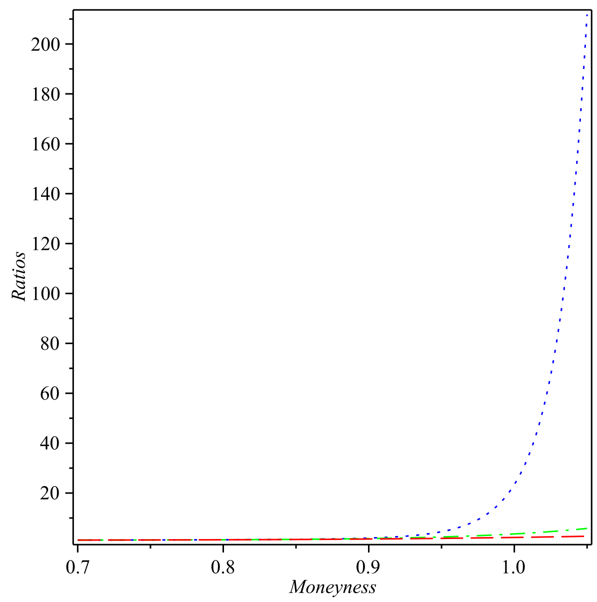


(b)

Figure 6.64: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

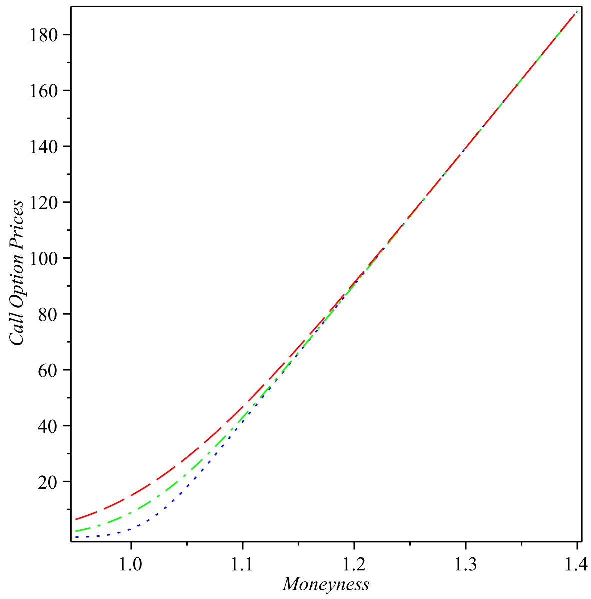


(a)

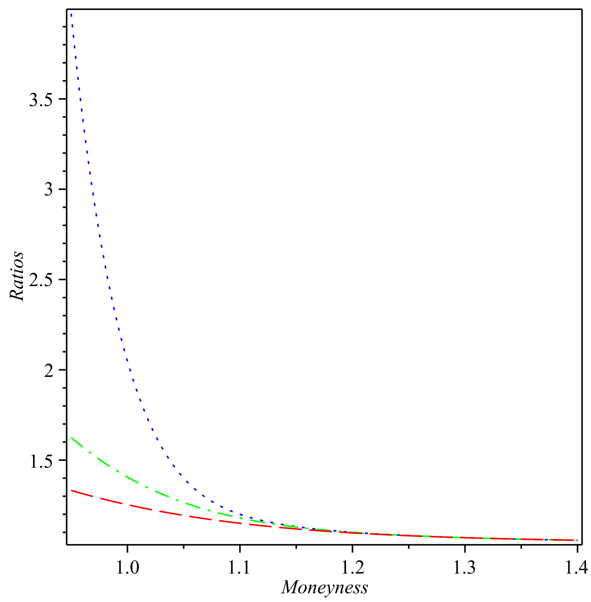


(b)

Figure 6.65: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

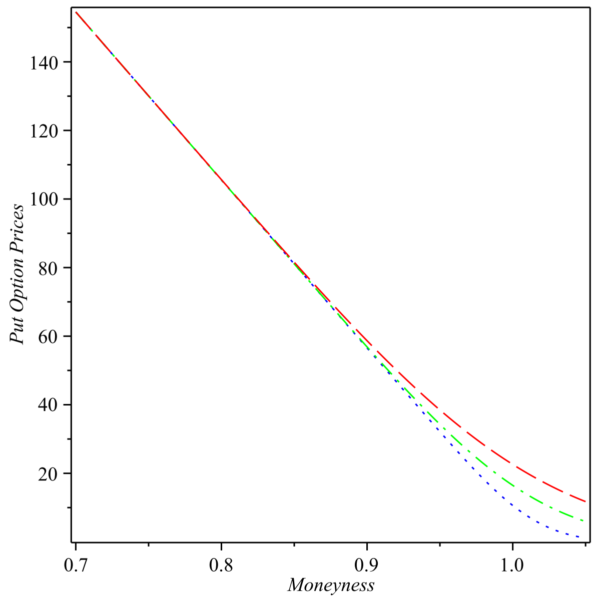


(a)

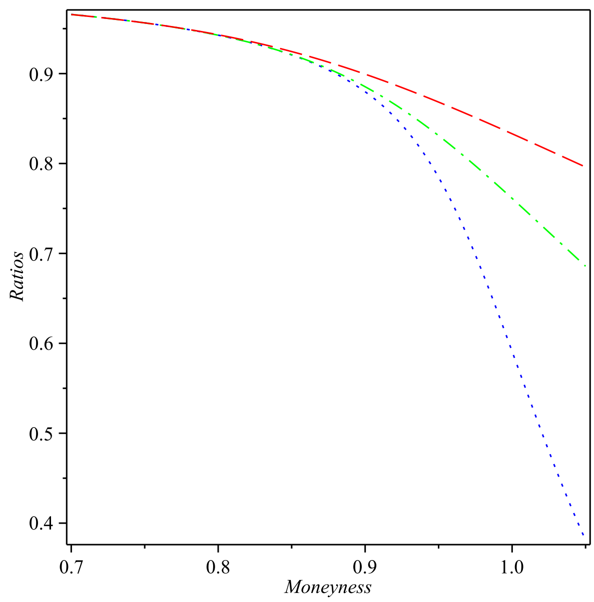


(b)

Figure 6.66: Sensitivity of call option prices to the stock volatility σ_S for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).



(a)



(b)

Figure 6.67: Sensitivity of put option prices to the stock volatility σ_S for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_S = 0.1$ (dotted line), $\sigma_S = 0.2$ (dash-dotted line), $\sigma_S = 0.3$ (dashed line).

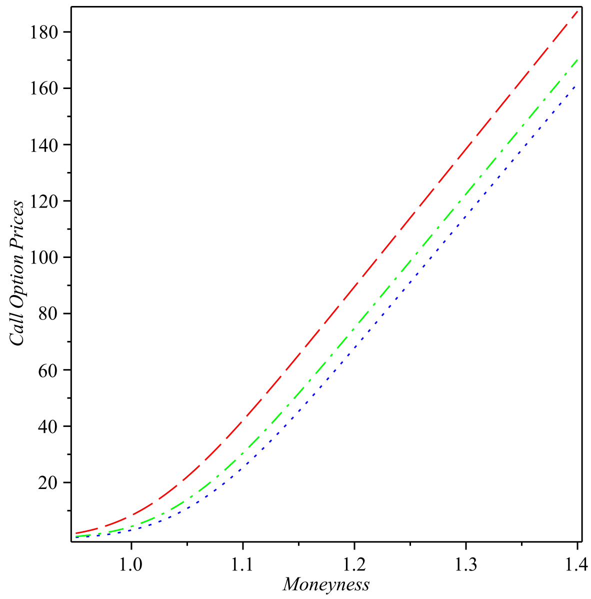
One of the standing features that we can observe from the results is for the in-the-money and the deep-in-the-money options: the changing of stock volatility almost has no impact. This feature is similar to that shown by Lioui's model results [38]. However due to the strong performance of stochastic property of our extended model, the option prices can be either overvalued or undervalued for both call and put options. Obviously, when the options are at-the-money or out-of-the-money or deep-out-of-the-money, the results show a high impact of the stock volatility on the option prices, which is similar to that found by Lioui's model [38]. The difference of our model is that any level of stock volatility may create either situation of overpriced and underpriced options, which implies that our stochastic earning yield option pricing model has a solid character of stochastic performance.

6.2.6 Dividend Yield Volatility (σ_{δ_i})

Because of uncertainty of dividend yield flow of the company, the volatility of the dividend yield (σ_{δ_i}) can be expected to play an important role in option pricing. By applying the same approach as previous sections, we test the sensitivity of option prices to the dividend yield volatility by changing the value of this parameter at three levels: $\sigma_{\delta_i} = 0.05$, $\sigma_{\delta_i} = 0.10$ and $\sigma_{\delta_i} = 0.20$. Again by setting the parameter σ_{δ_i} at a value, we mean that we set the same value for $i = 1, 2$ and 3. For example, $\sigma_{\delta_i} = 0.05$ means $\sigma_{\delta_1} = 0.05$, $\sigma_{\delta_2} = 0.05$ and $\sigma_{\delta_3} = 0.05$ for the simplicity of the test. We set the other parameters for simulation at the values shown in table 6.8. The simulation results are shown in Figures 6.68 - 6.85.

Table 6.8: Parameter values used in the sensitivity analysis of the model with respect to the dividend yield volatility

$T = 0.1$	$\sigma_{\xi 1} = 0.05$	$\theta_{\kappa 1} = 0.45$
σ_S=0.05	$\sigma_{\xi 2} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\xi 3} = 0.05$	$\theta_{\kappa 3} = 0.45$
		$\sigma_{\kappa 1} = 0.10$
		$\sigma_{\kappa 1} = 0.10$
		$\sigma_{\kappa 1} = 0.10$



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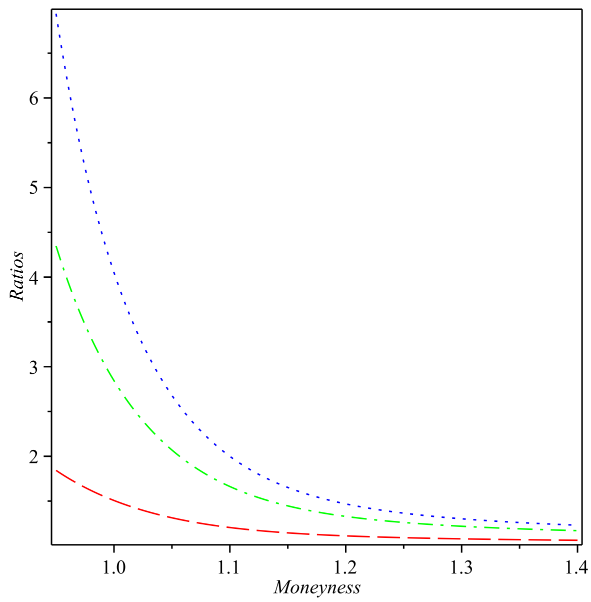
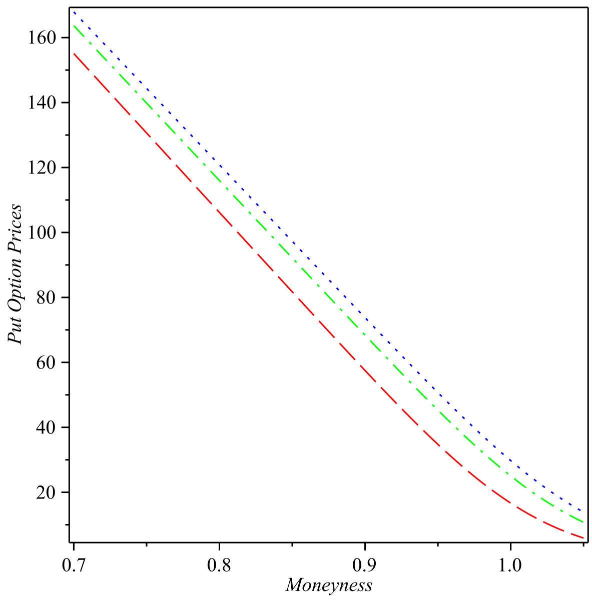
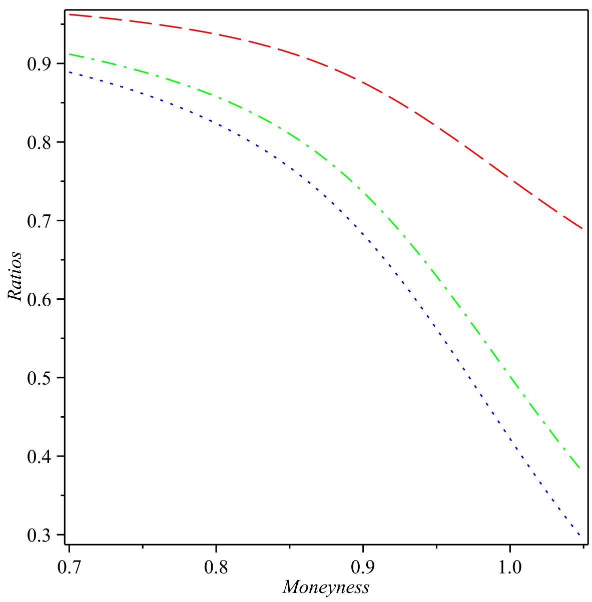


(b)

Figure 6.68: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).

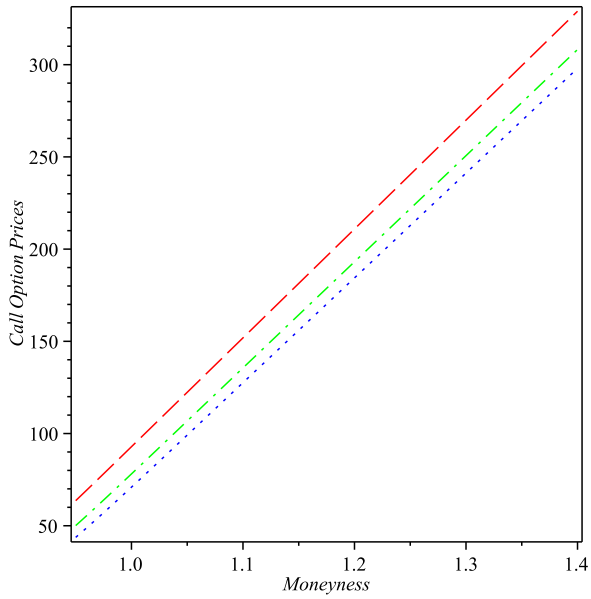


(a)



(b)

Figure 6.69: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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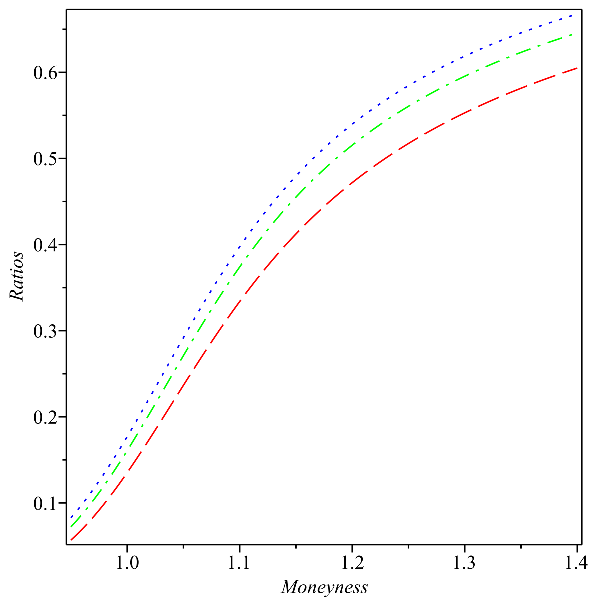
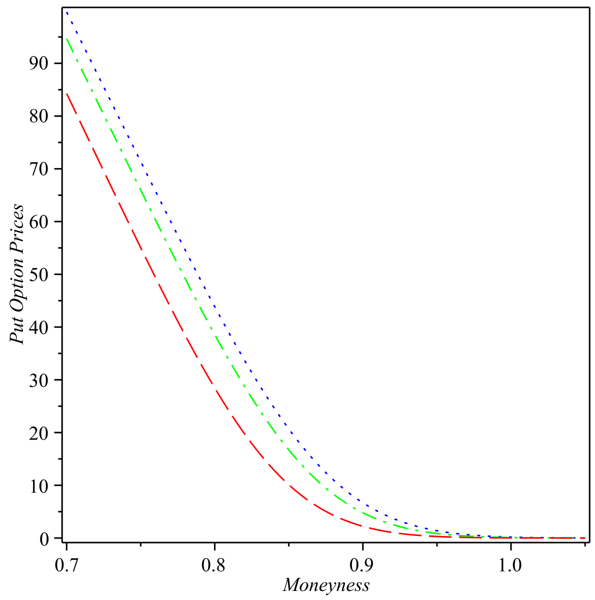


(b)

Figure 6.70: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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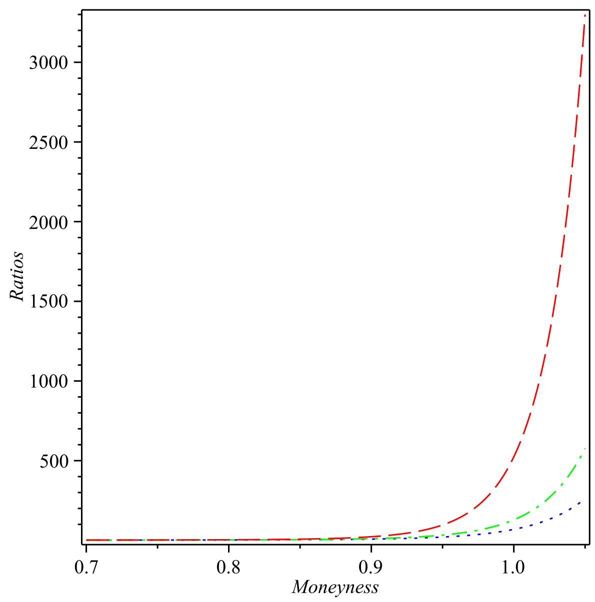
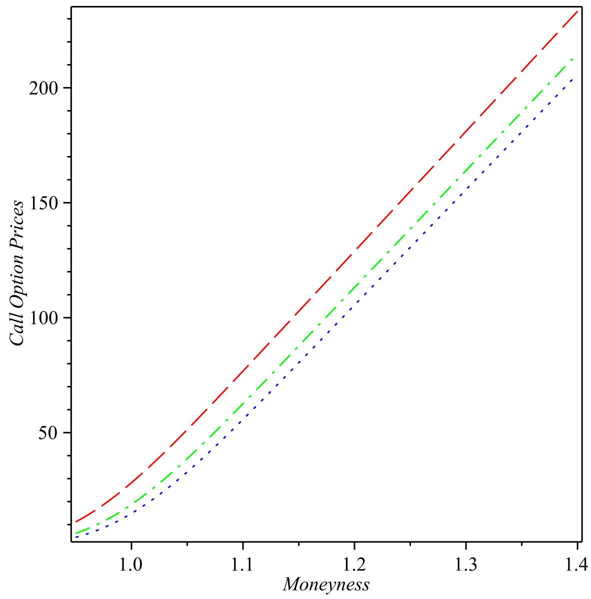


(b)

Figure 6.71: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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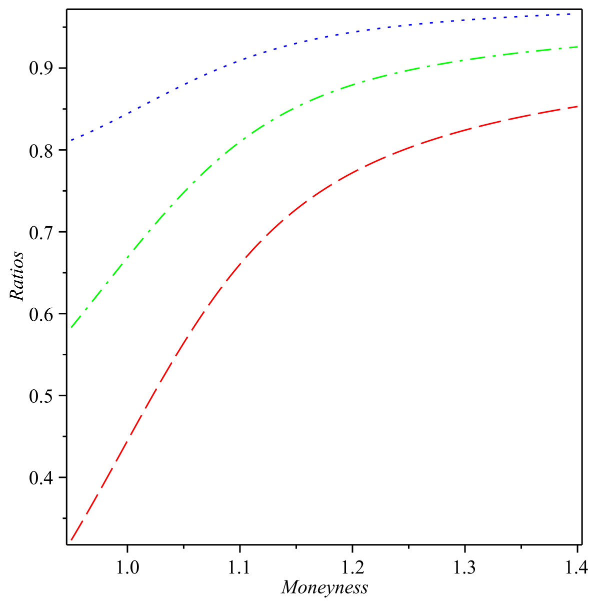
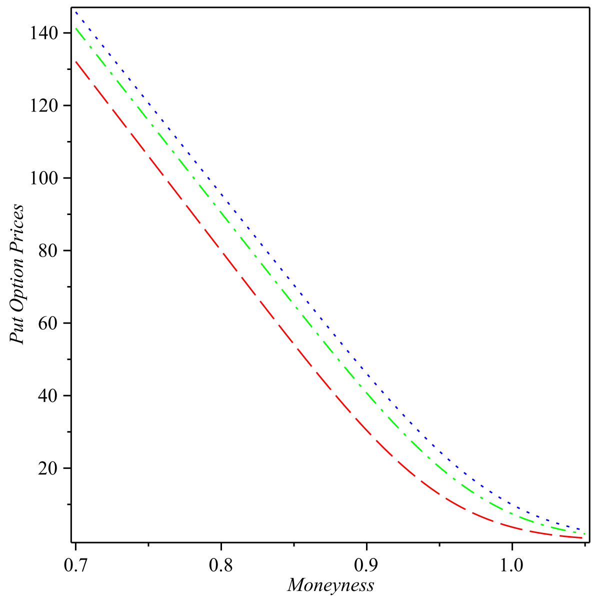


(b)

Figure 6.72: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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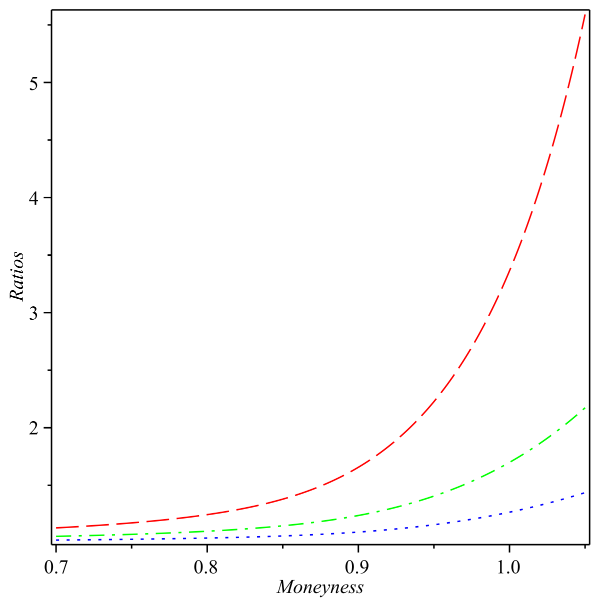
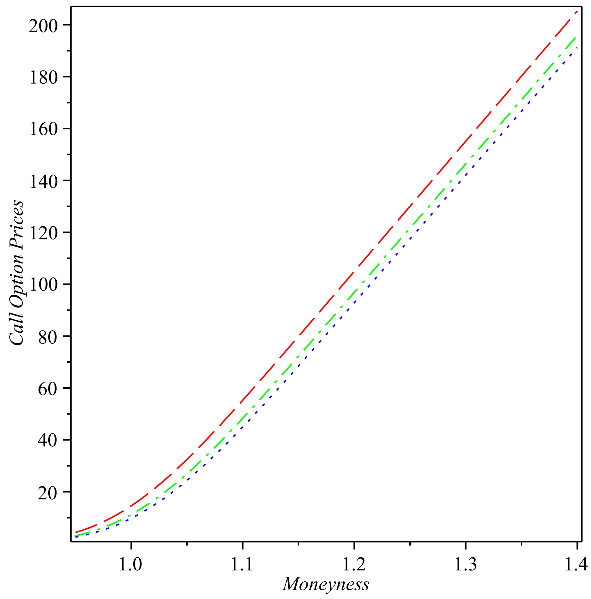


(b)

Figure 6.73: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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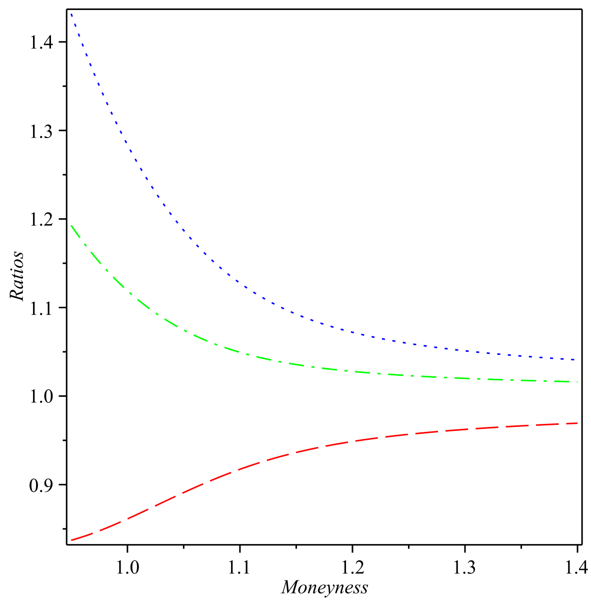
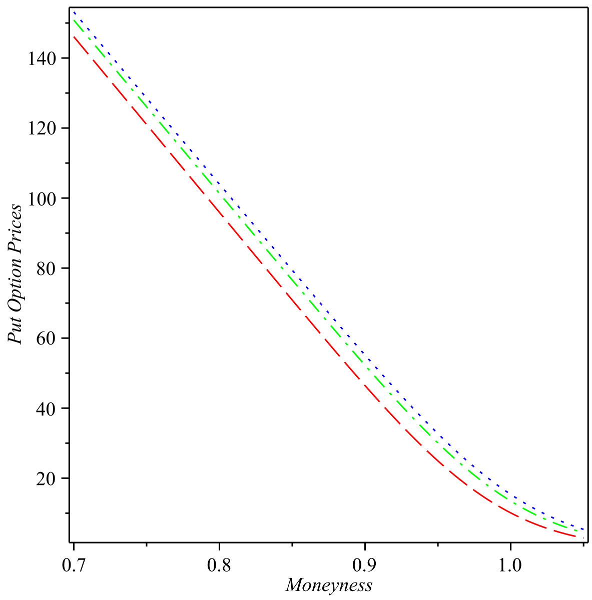


(b)

Figure 6.74: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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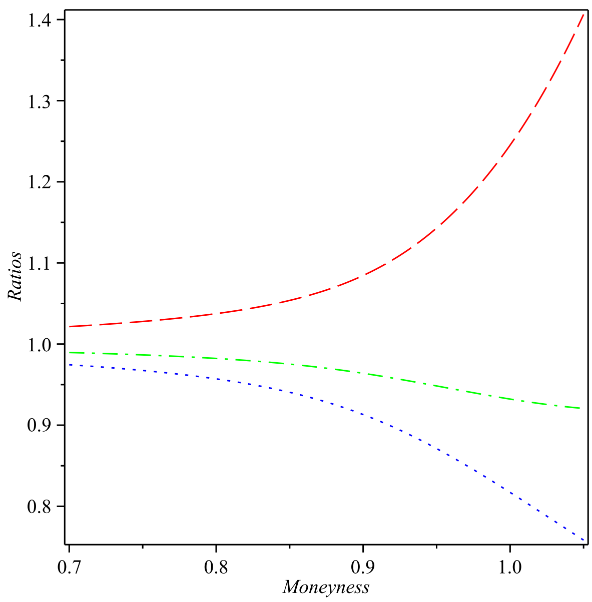
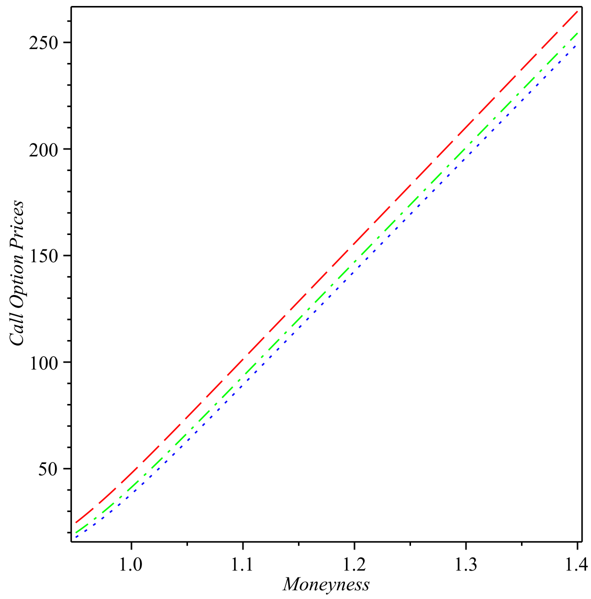


(b)

Figure 6.75: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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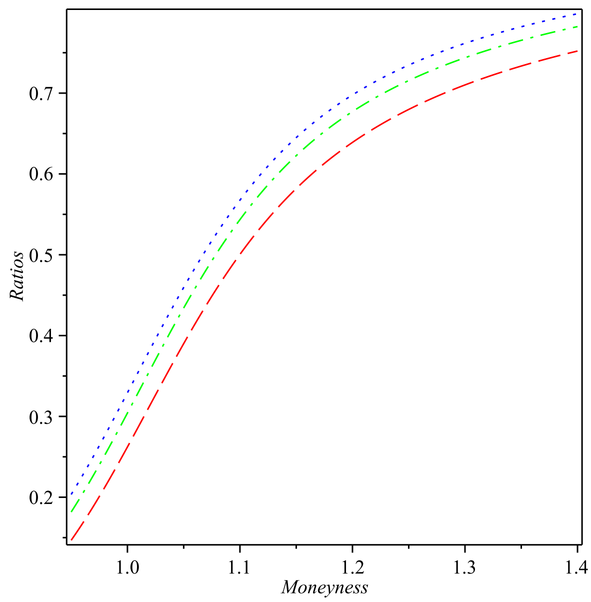
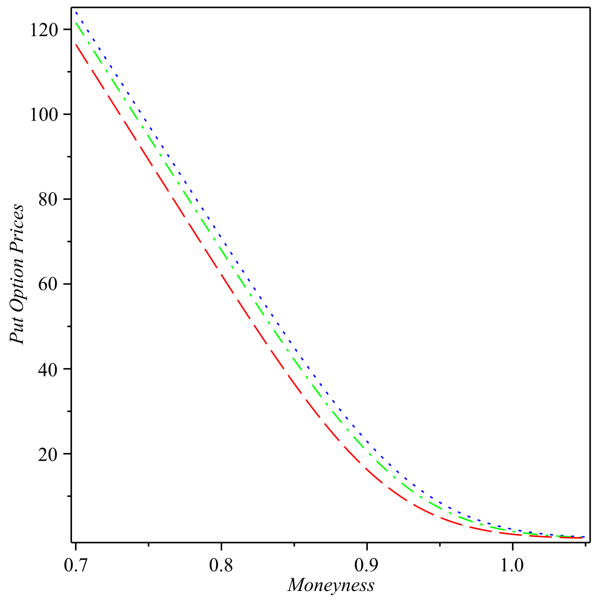


(b)

Figure 6.76: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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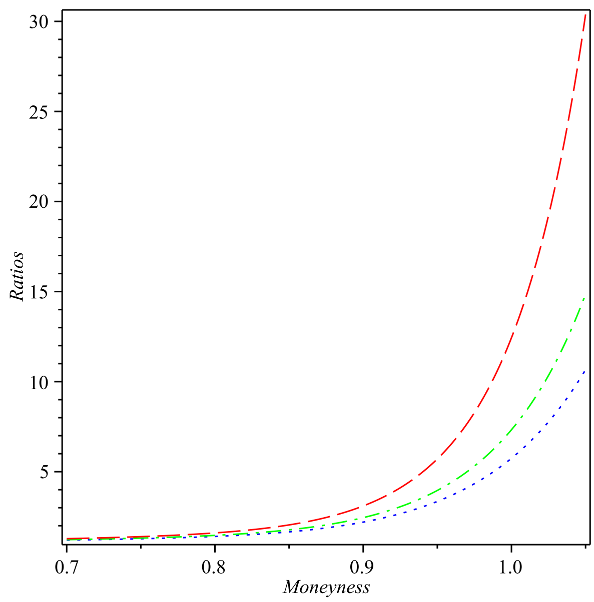
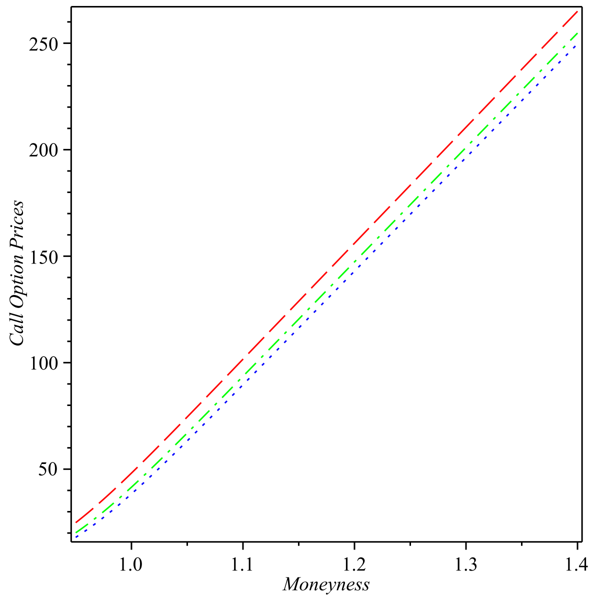


(b)

Figure 6.77: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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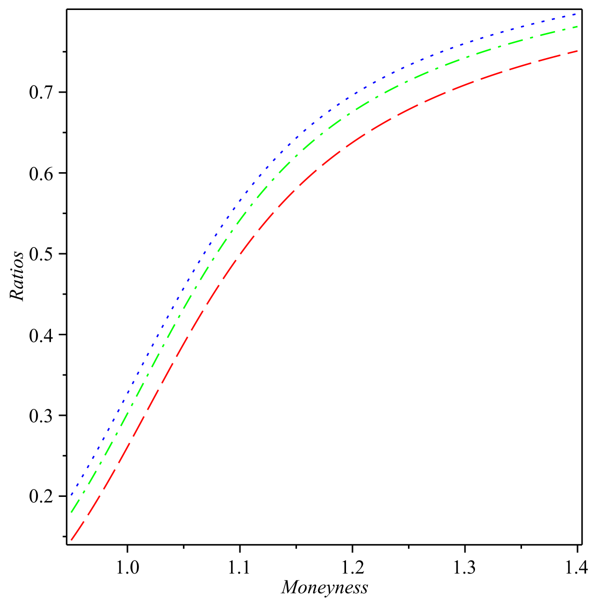
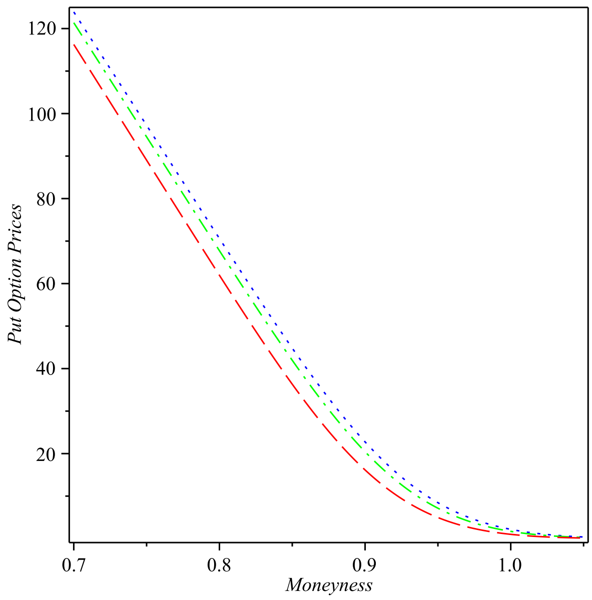


(b)

Figure 6.78: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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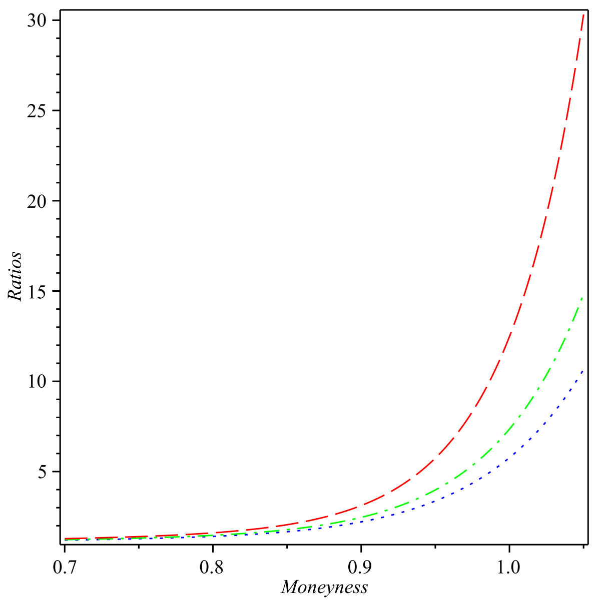
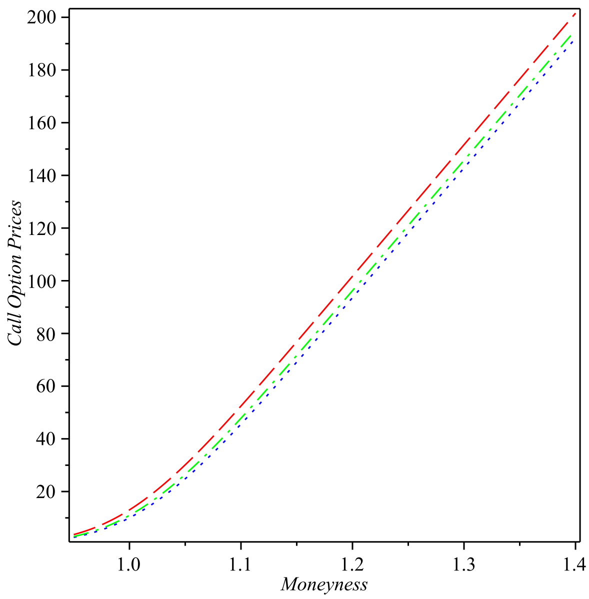


(b)

Figure 6.79: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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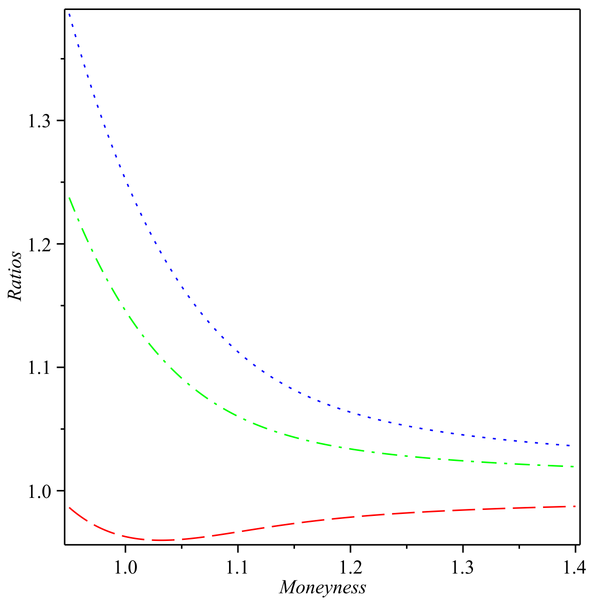
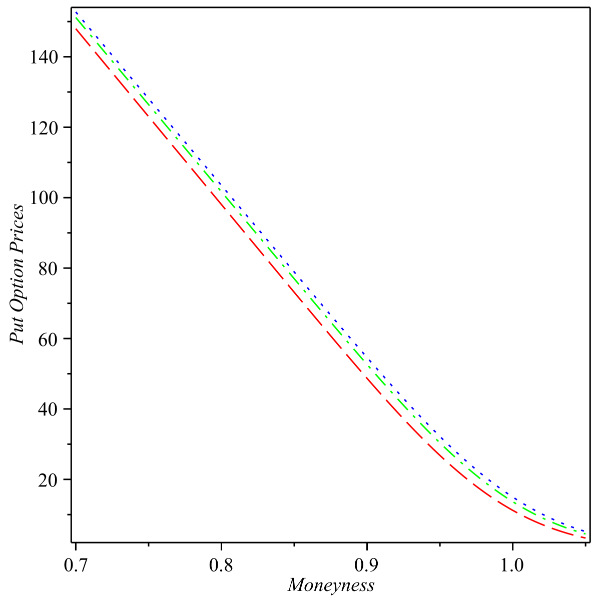


(b)

Figure 6.80: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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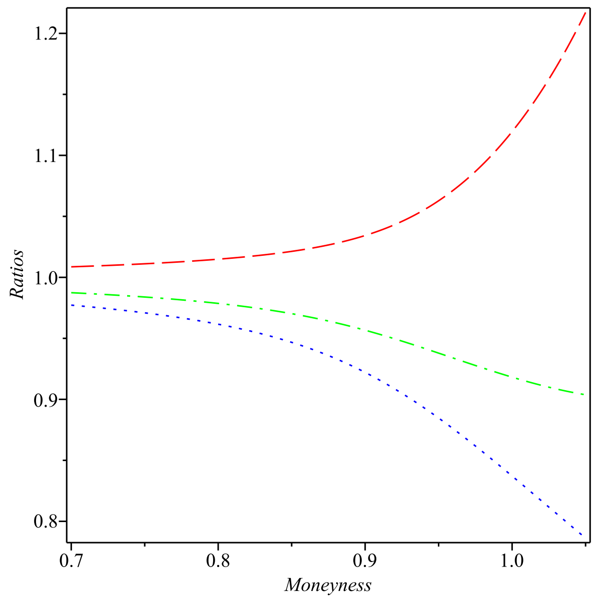

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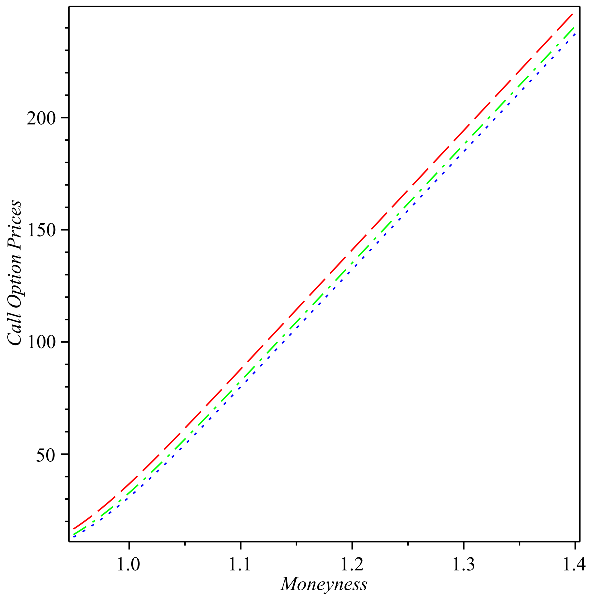
Figure 6.81: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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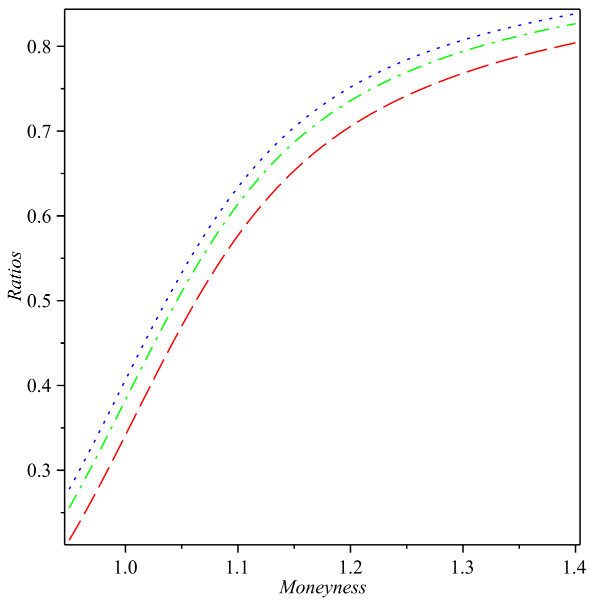

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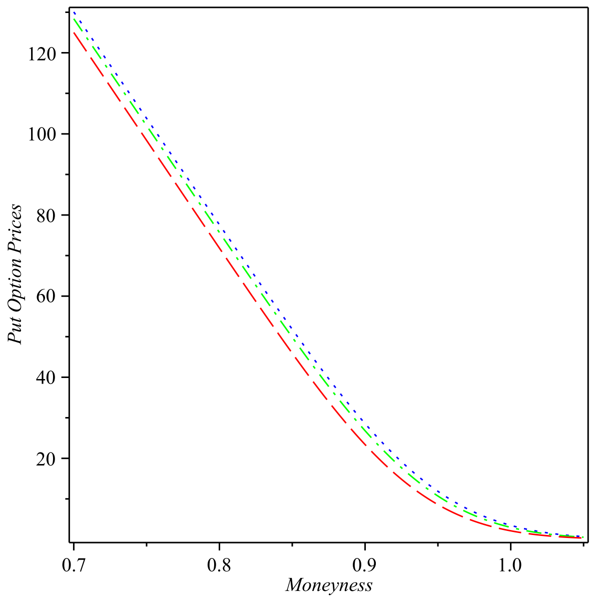
Figure 6.82: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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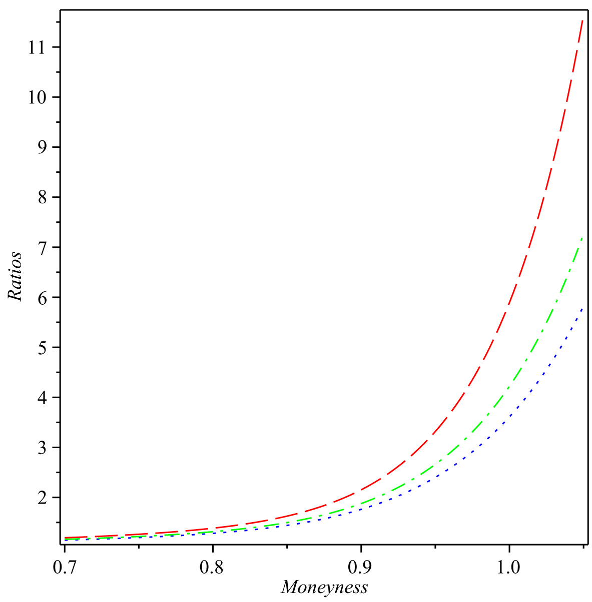
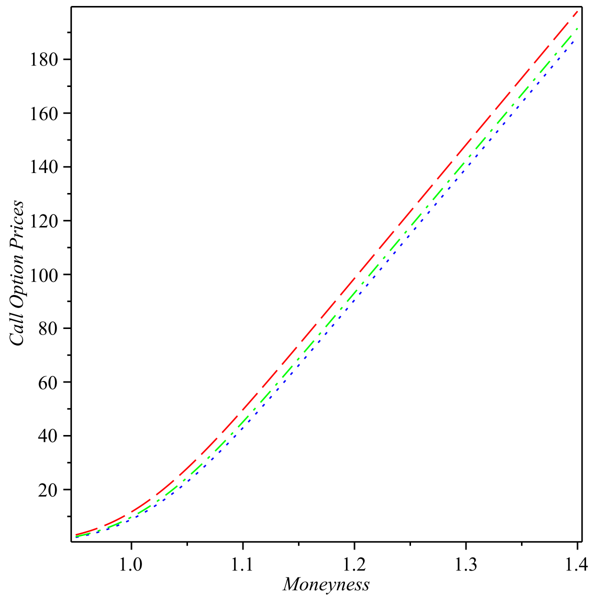


(b)

Figure 6.83: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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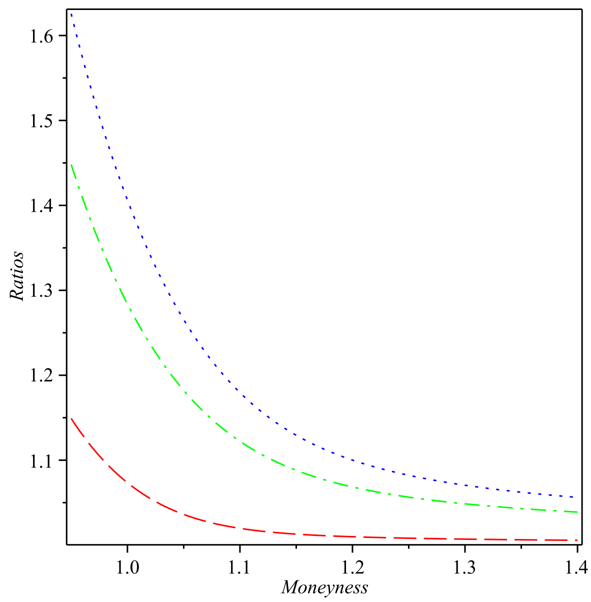
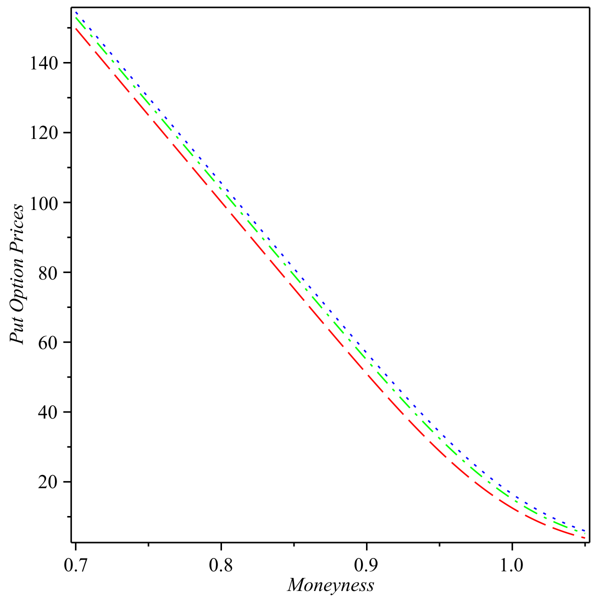


(b)

Figure 6.84: Sensitivity of call option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).



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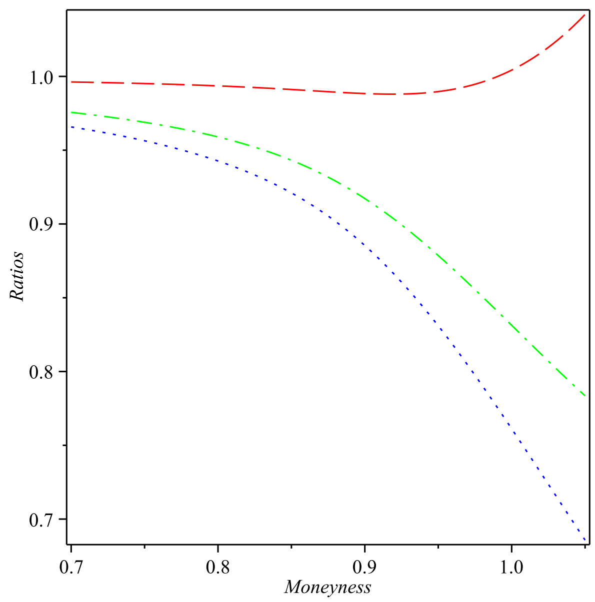

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Figure 6.85: Sensitivity of put option prices to the dividend yield volatility σ_{δ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\delta_i} = 0.05$ (dotted line), $\sigma_{\delta_i} = 0.10$ (dash-dotted line), $\sigma_{\delta_i} = 0.20$ (dashed line).

The simulation results have shown that for most of the deep-out-of-the-money options, the change of dividend yield volatility has a high impact on the option prices for both call and put options. While in-the-money and deep-in-the-money options might derive some impacts or some slight impacts when changing the level of dividend yield volatility.

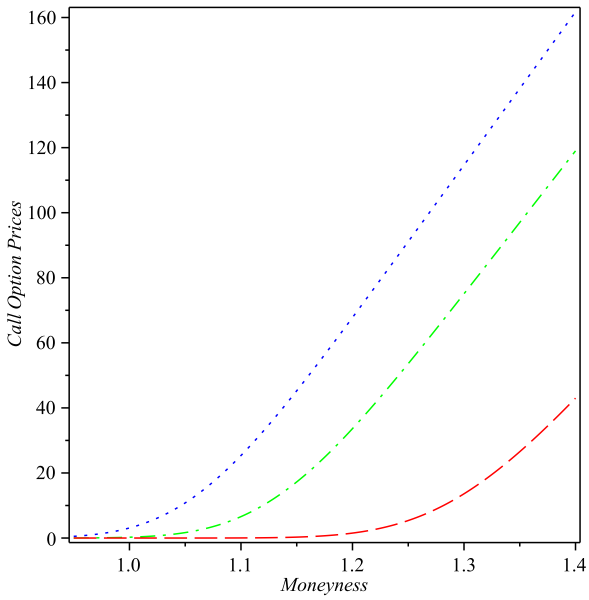
6.2.7 Earning Yield Volatility (σ_{ξ_i})

In this thesis, we extend the option pricing model by taking into account the earning yield parameter. Consequently, earning yield volatility is a parameter which should be analyzed for the sensitivity for the option prices. The same approach is applied in this section in order to test the sensitivity of option prices to the earning yield volatility (σ_{ξ_i}). By setting three levels of the parameter: $\sigma_{\xi_i} = 0.05$, $\sigma_{\xi_i} = 0.10$ and $\sigma_{\xi_i} = 0.20$, we run the simulation for the sensitivity test under various different circumstances of friction market. As before, for the σ_{ξ_i} from three different sources, we will set the same value for the parameter in all sources ($i = 1, 2$ and 3). The other parameters are set at values as shown in Table 6.9.

Table 6.9: Parameter values used in the sensitivity analysis of the model with respect to the earning yield volatility

$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\theta_{\kappa 1} = 0.45$
σ_S=0.05	$\sigma_{\delta 2} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\delta 3} = 0.05$	$\theta_{\kappa 3} = 0.45$
		$\sigma_{\kappa 1} = 0.10$
		$\sigma_{\kappa 1} = 0.10$
		$\sigma_{\kappa 1} = 0.10$

The results of simulations under different combination of θ_δ and θ_ξ are shown in figures 6.86-6.103.



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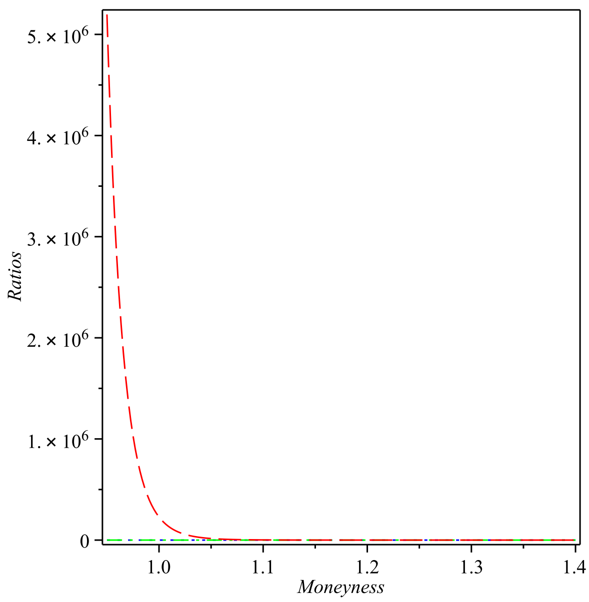
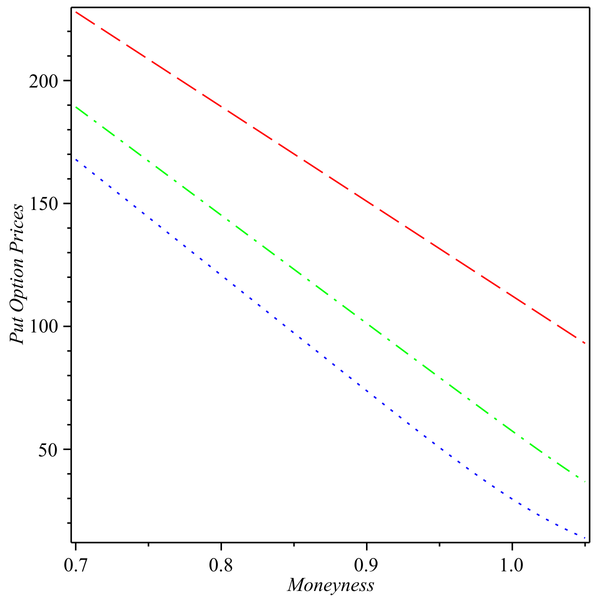


(b)

Figure 6.86: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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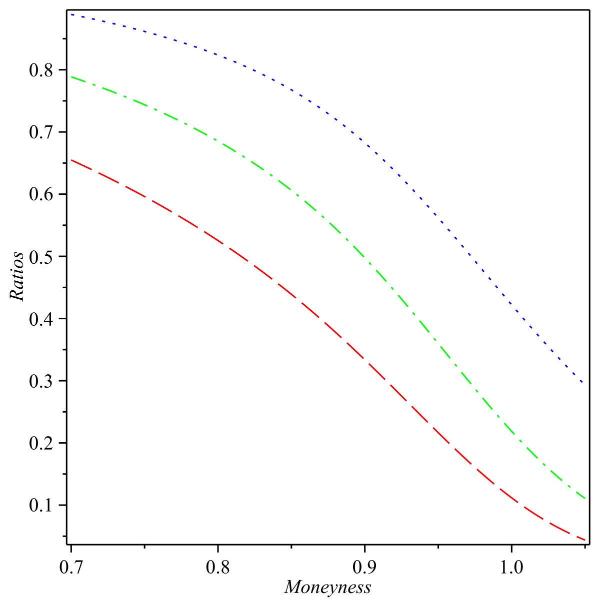

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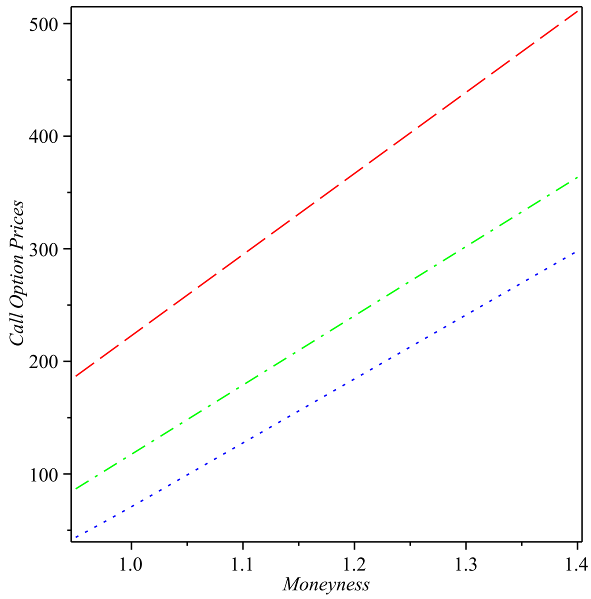
Figure 6.87: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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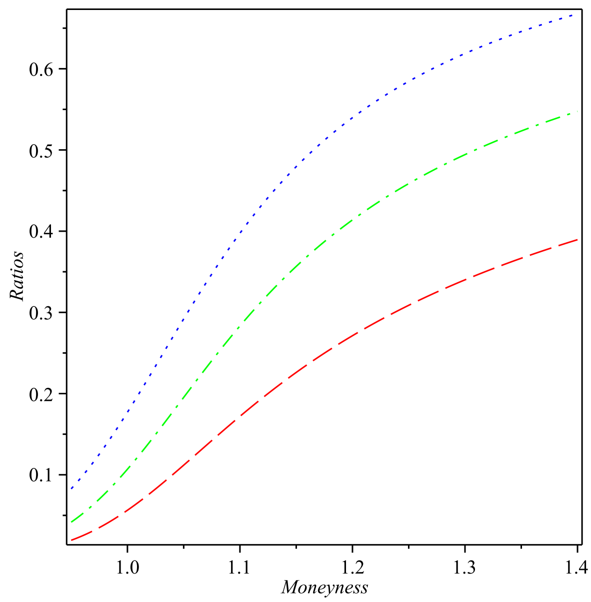

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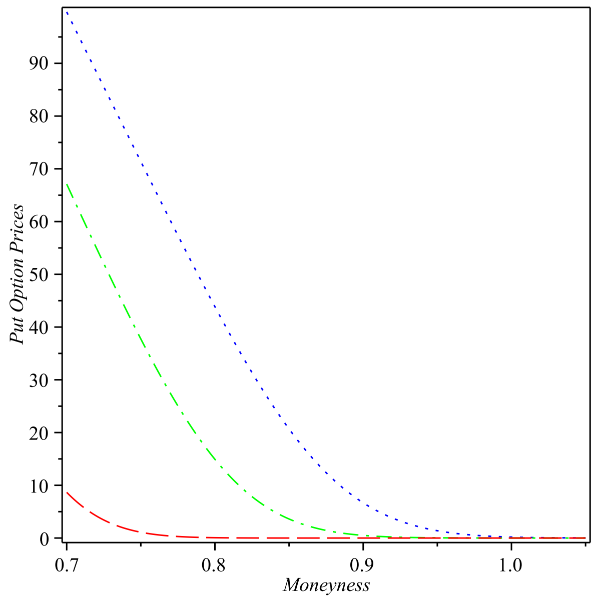
Figure 6.88: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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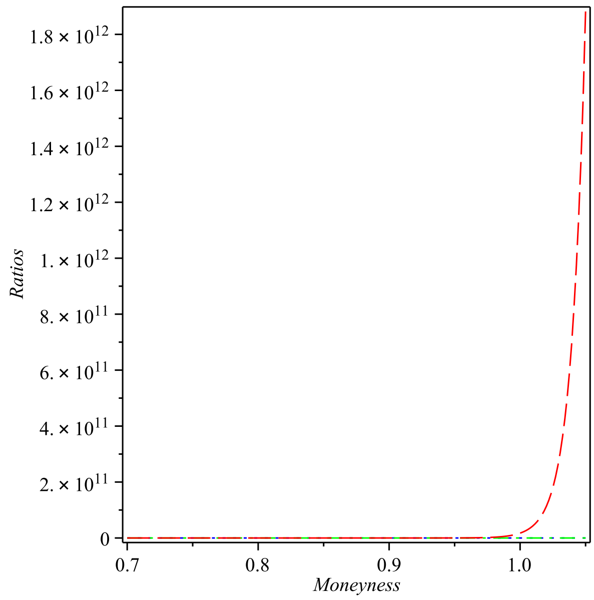

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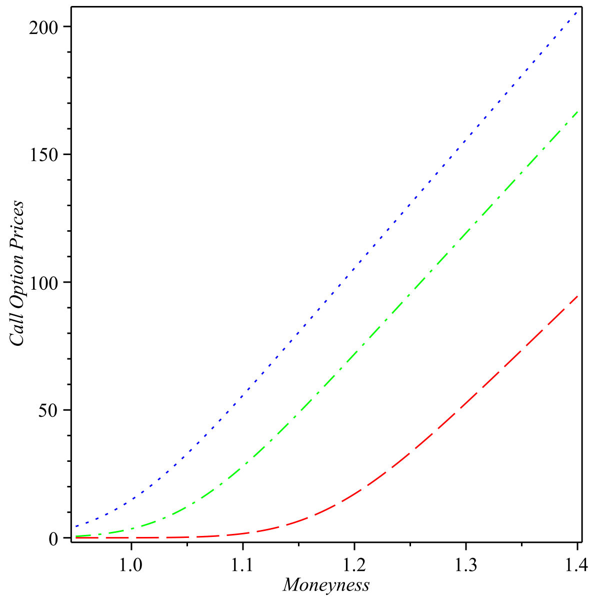
Figure 6.89: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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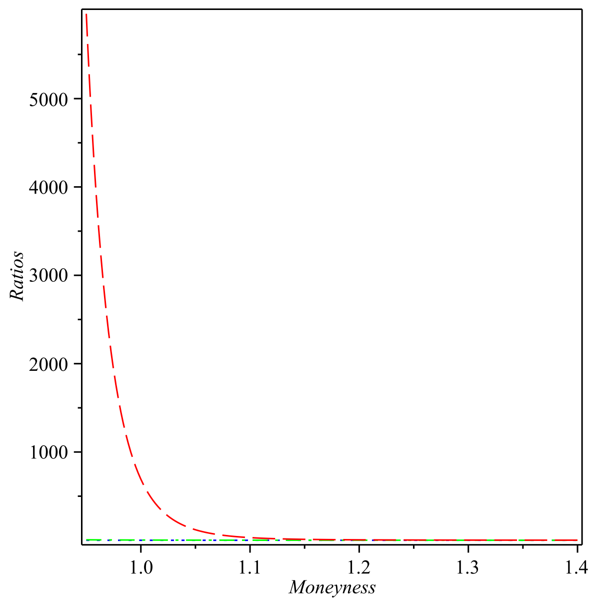
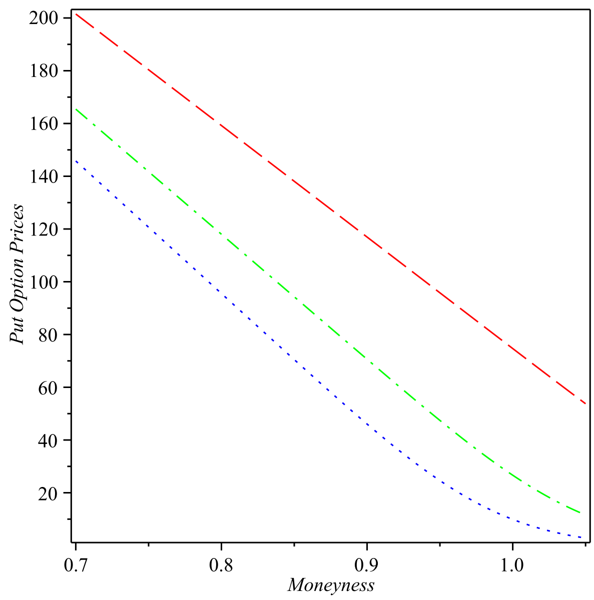


(b)

Figure 6.90: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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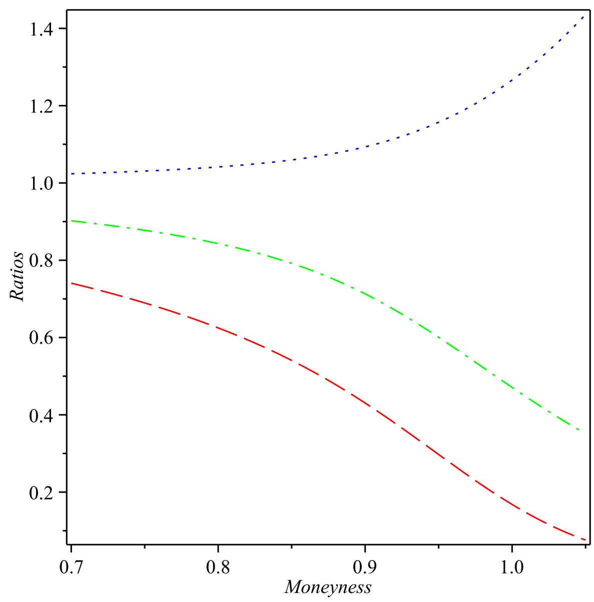

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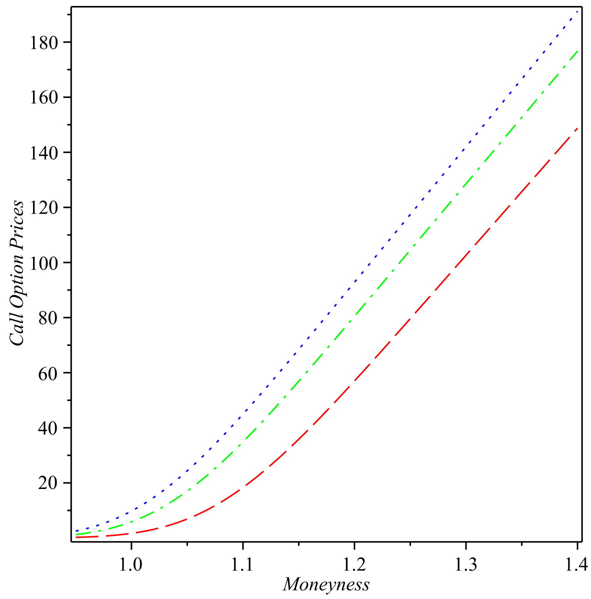
Figure 6.91: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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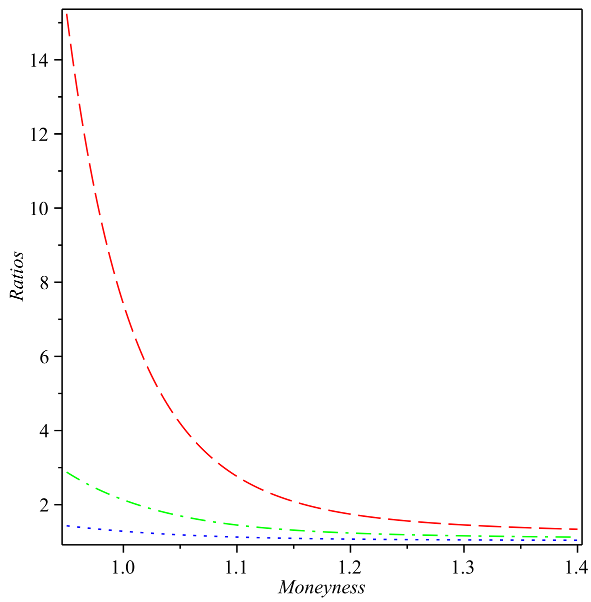

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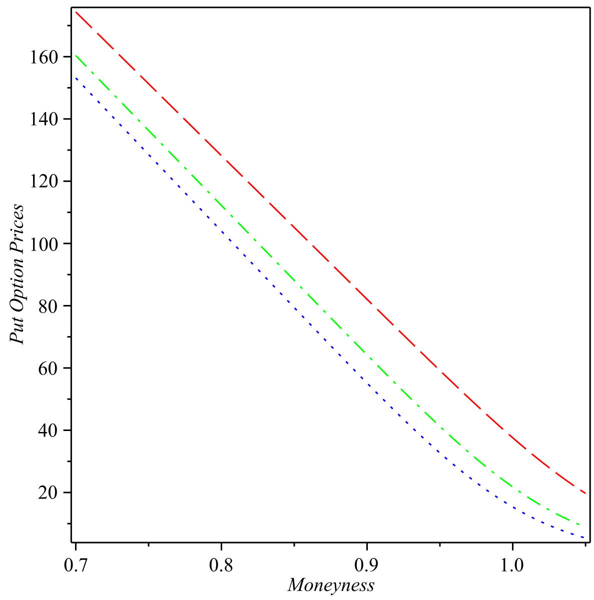
Figure 6.92: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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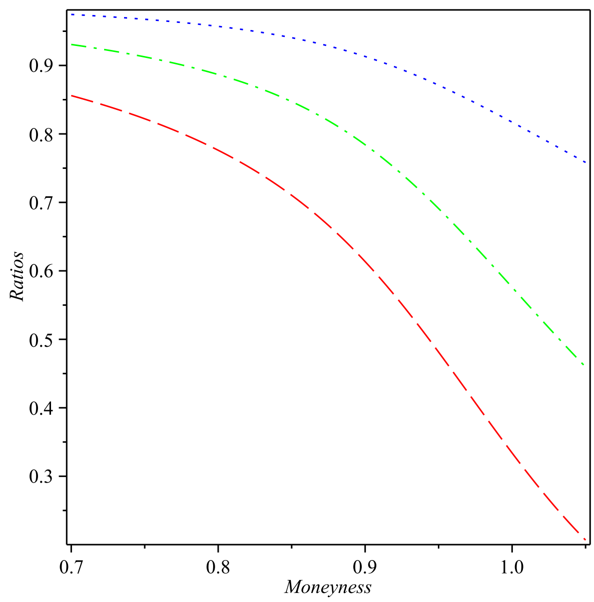
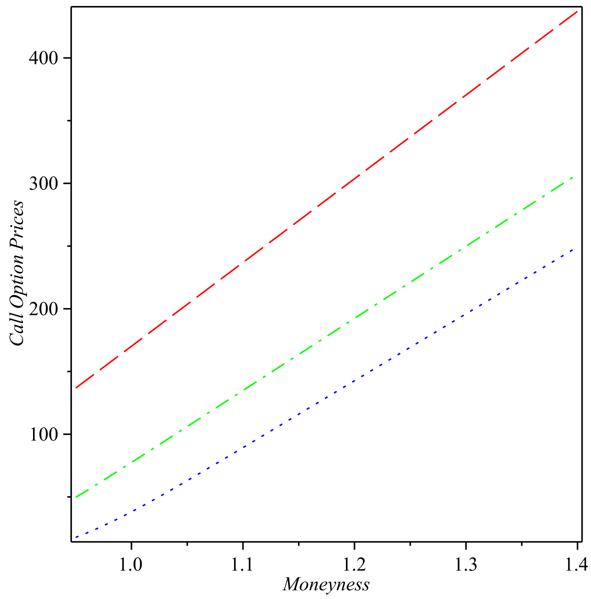


(b)

Figure 6.93: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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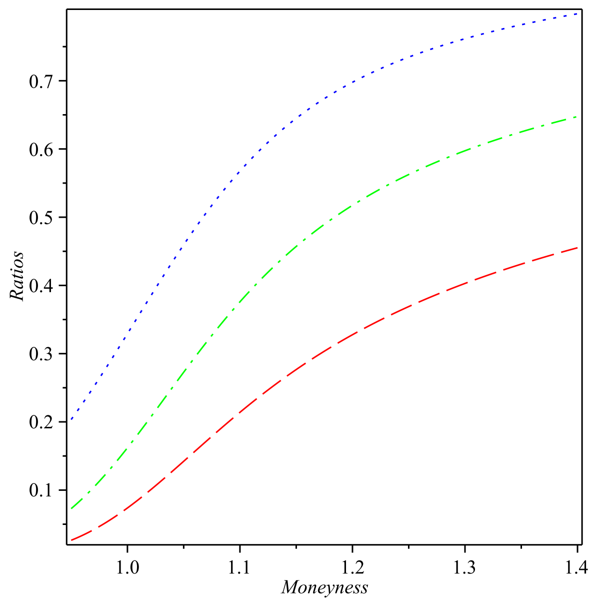
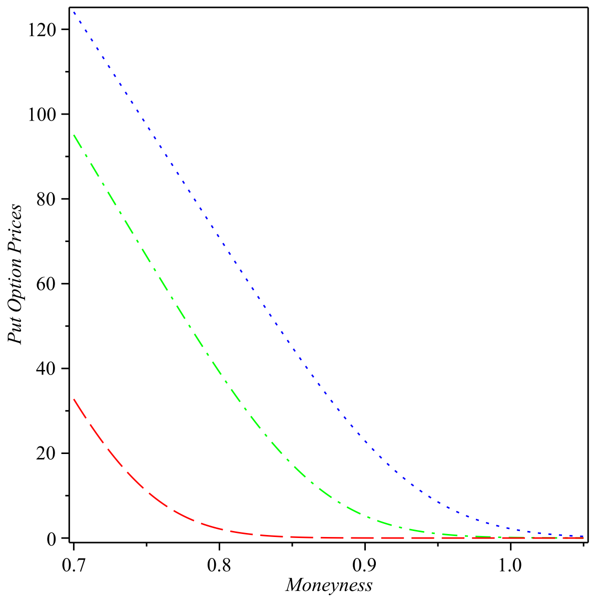


(b)

Figure 6.94: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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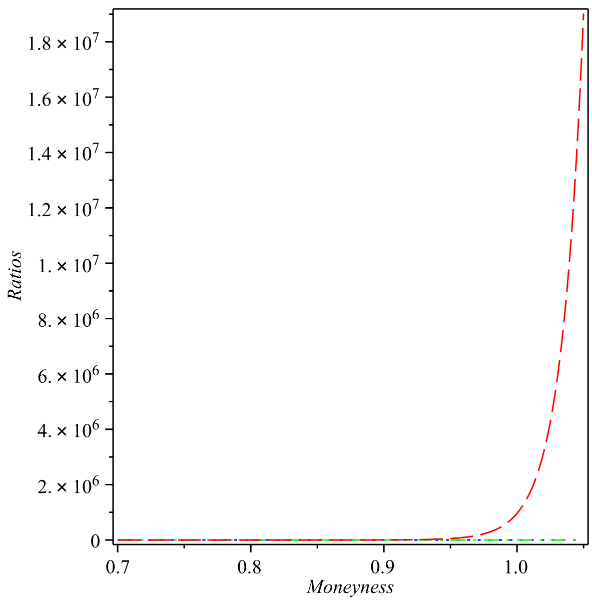

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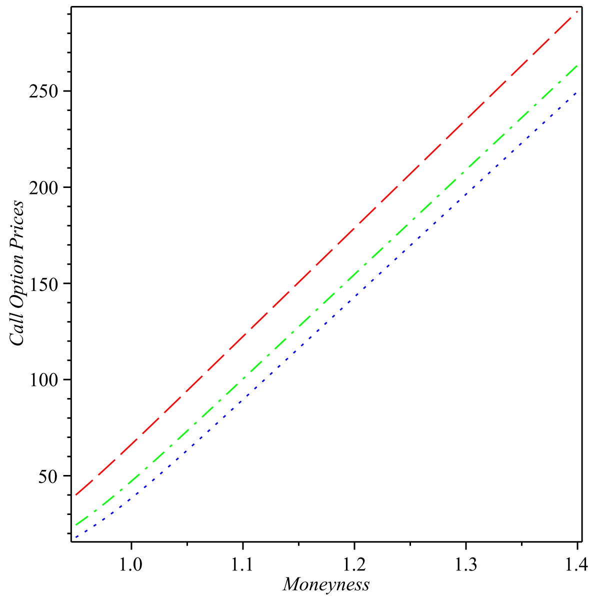
Figure 6.95: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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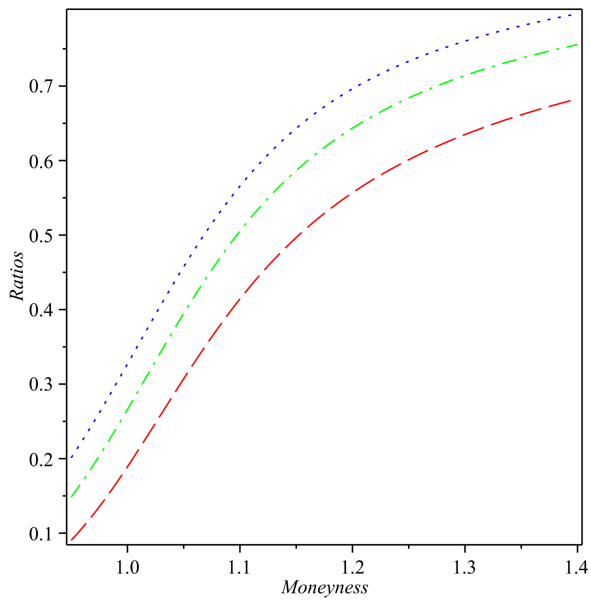

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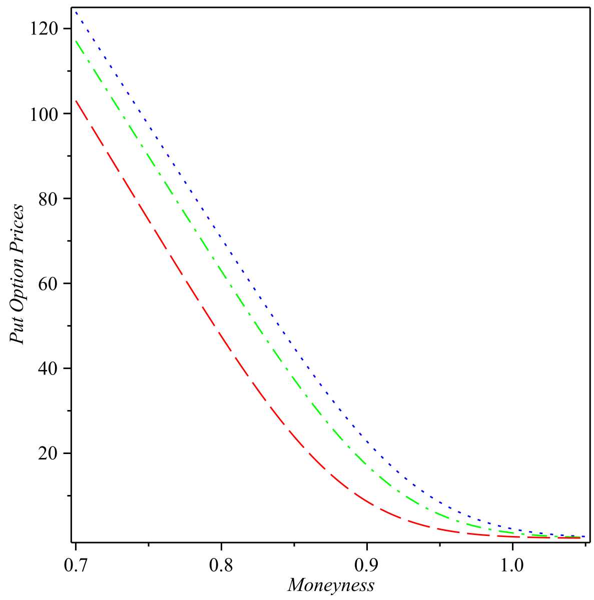
Figure 6.96: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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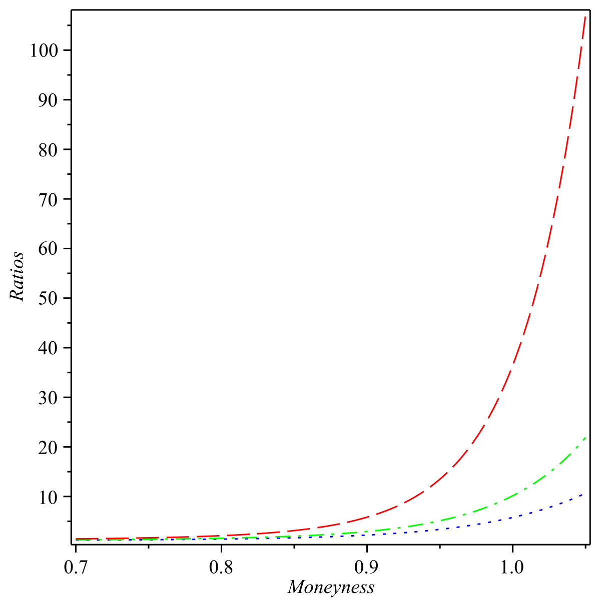

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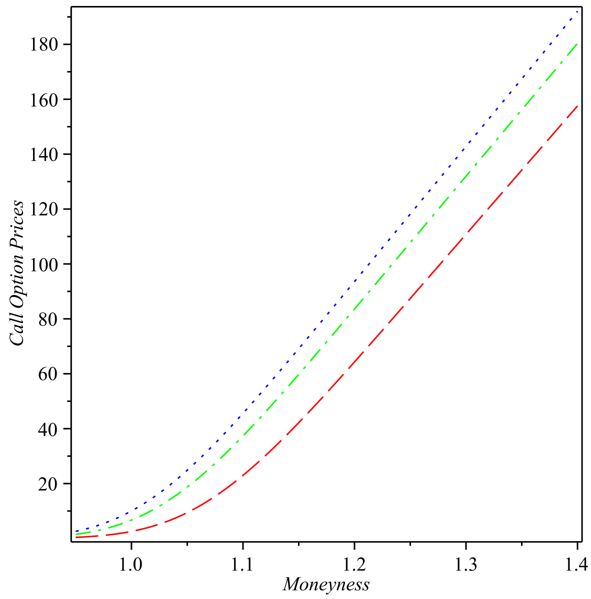
Figure 6.97: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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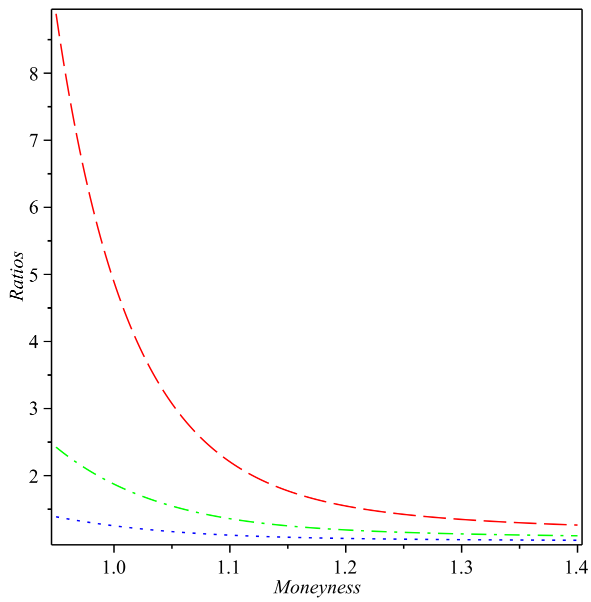
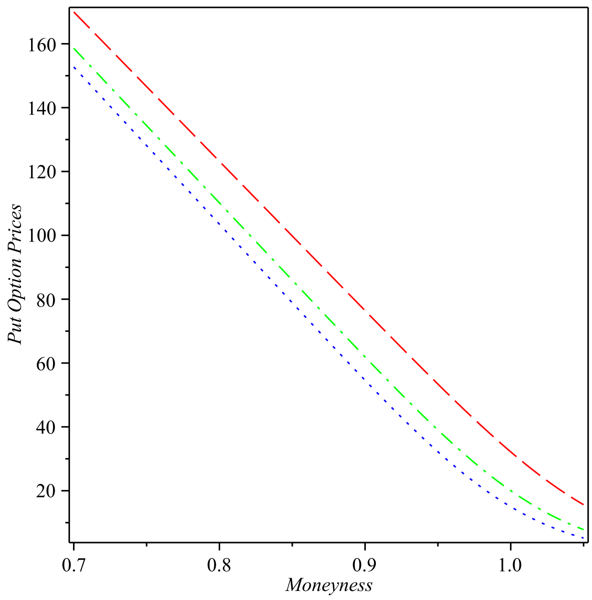


(b)

Figure 6.98: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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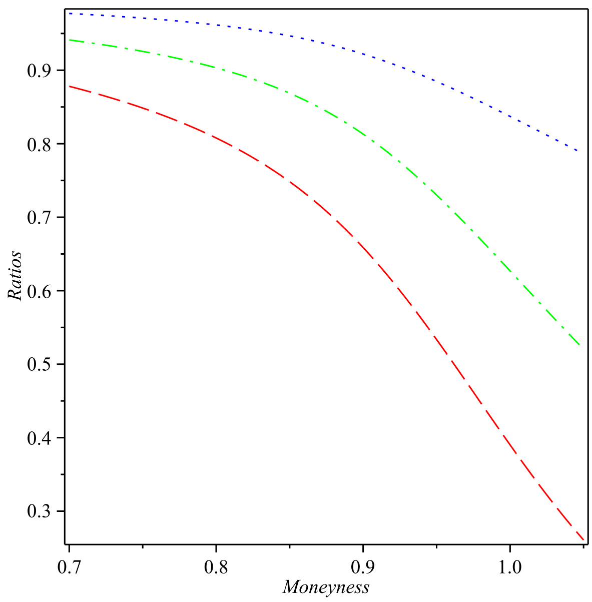

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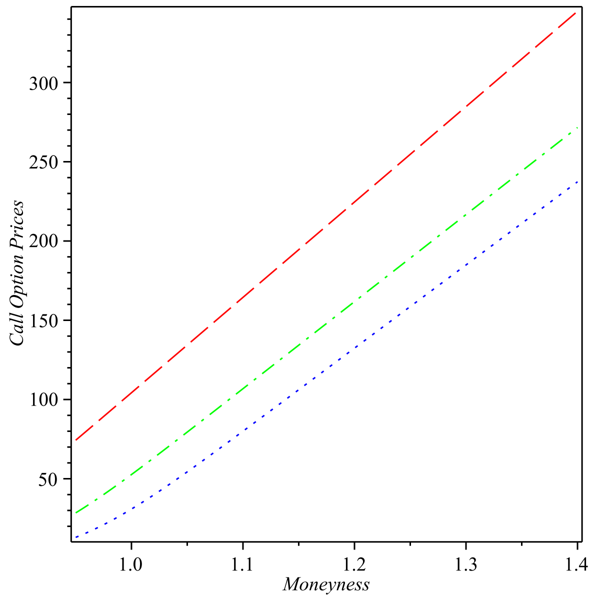
Figure 6.99: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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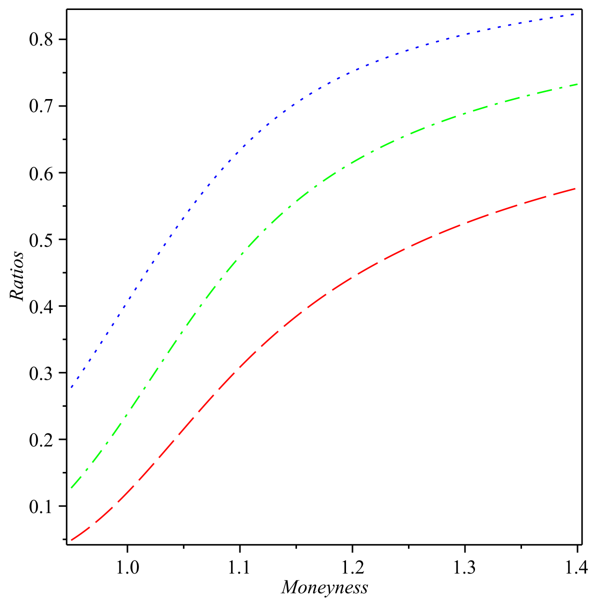
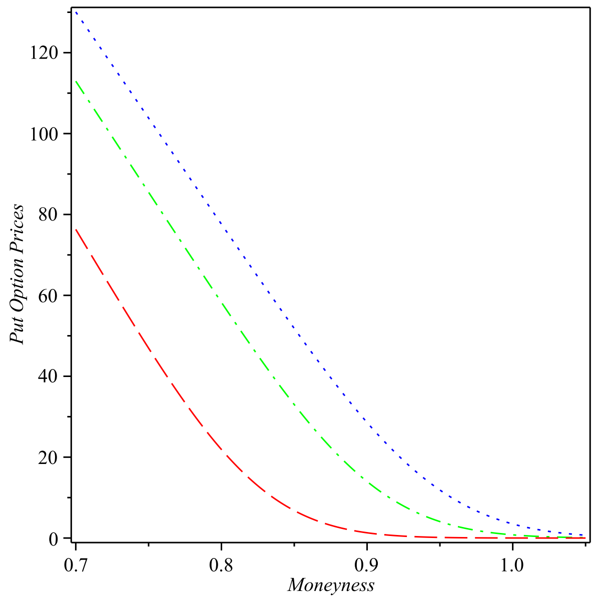


(b)

Figure 6.100: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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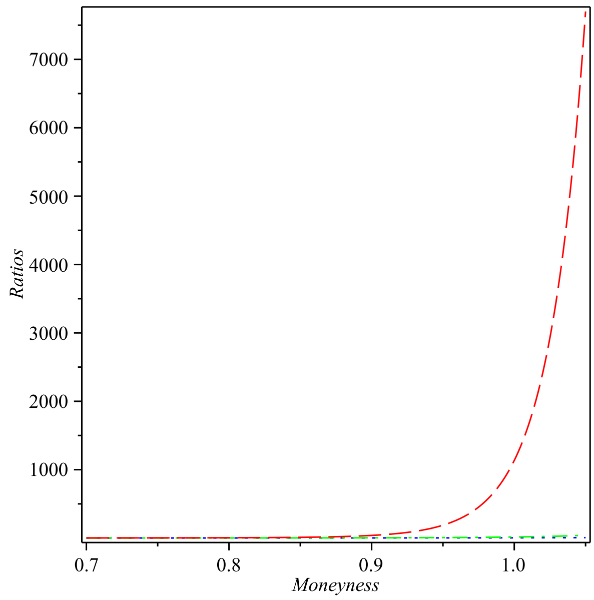
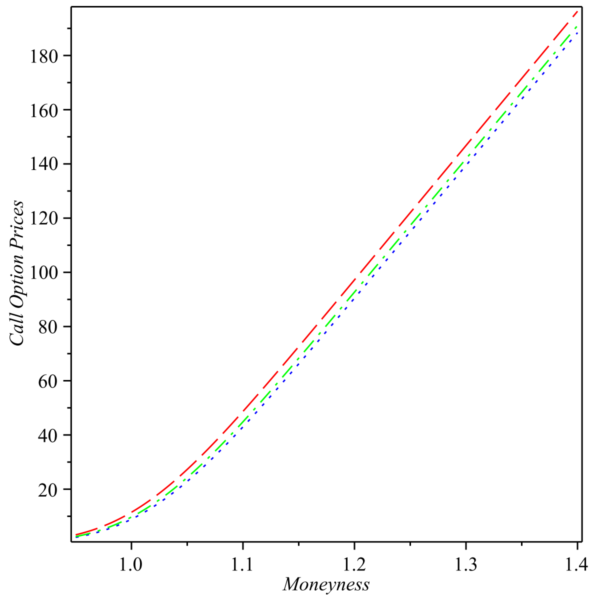


(b)

Figure 6.101: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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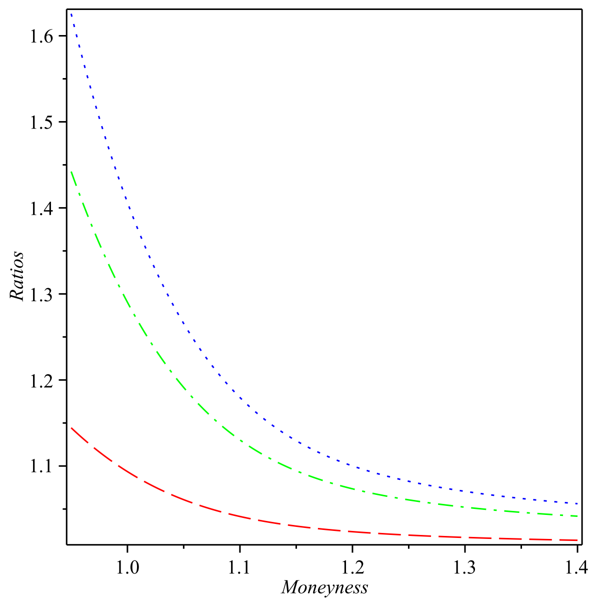
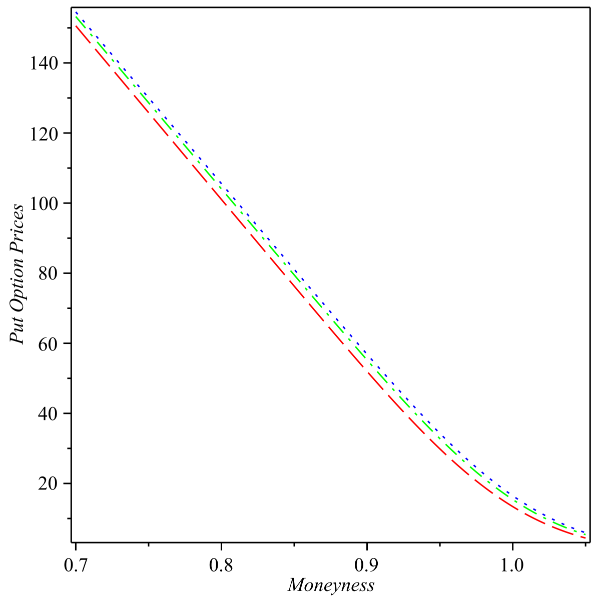


(b)

Figure 6.102: Sensitivity of call option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).



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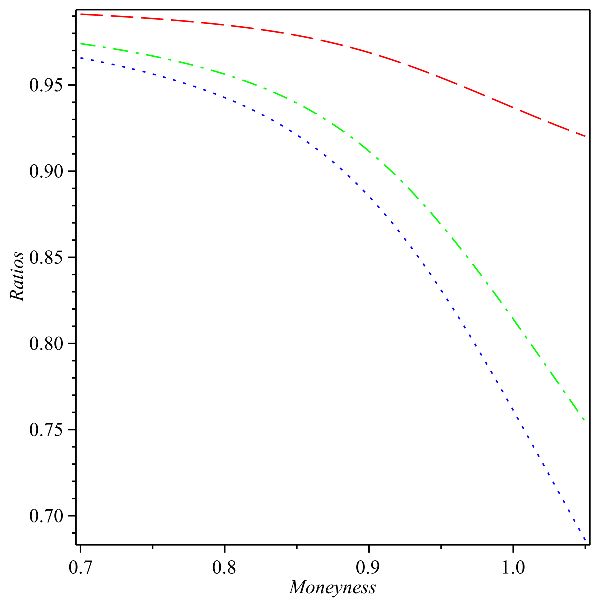


(b)

Figure 6.103: Sensitivity of put option prices to the earning yield volatility σ_{ξ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\xi_i} = 0.05$ (dotted line), $\sigma_{\xi_i} = 0.10$ (dash-dotted line), $\sigma_{\xi_i} = 0.20$ (dashed line).)

Figure 6.86 to 6.103 show the results as we expect. Since the earning yield is the process under the dividend yield, the results give similar behavior. Obviously, change of the earning yield volatility has impact on the option prices. Particularly, change of the earning yield volatility has a significant impact on the deep-out-of-the-money options. In some situations, the option price also changes drastically even when it is situated in the very-deep-out-of-the-money moneyness, for example when $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$ in figure 6.98 and 6.99. Similarly, the option prices can be either overpriced or underpriced depending on the market frictions.

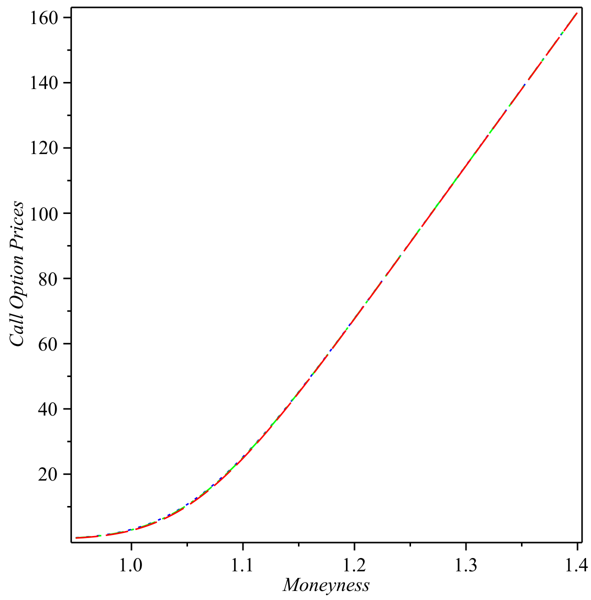
6.2.8 Market Price of Risk Volatility (σ_{κ_i})

The last parameter for sensitivity analysis is the volatility of market price of risk σ_{κ_i}. By applying the same approach, we test the influence of this parameter under various different circumstances. We vary the market price of risk volatility at three values: $\sigma_{\kappa_i}(MPRV) = 0.20$, $\sigma_{\kappa_i}(MPRV) = 0.50$ and $\sigma_{\kappa_i}(MPRV) = 0.90$ where $i = 1, 2$ and 3. The other parameters used for the sensitivity test of market price of risk volatility are set at the values as shown in the Table 6.10.

Table 6.10: Parameter values used in the sensitivity analysis of the model with respect to the market price of risk volatility

$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\theta_{\kappa 1} = 0.45$
σ_S=0.05	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\theta_{\kappa 2} = 0.45$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 3} = 0.05$	$\theta_{\kappa 3} = 0.45$

Figures 6.104-6.121 show the results of simulation for investigating the sensitivity of option prices to the market price of risk volatility under various market situations.



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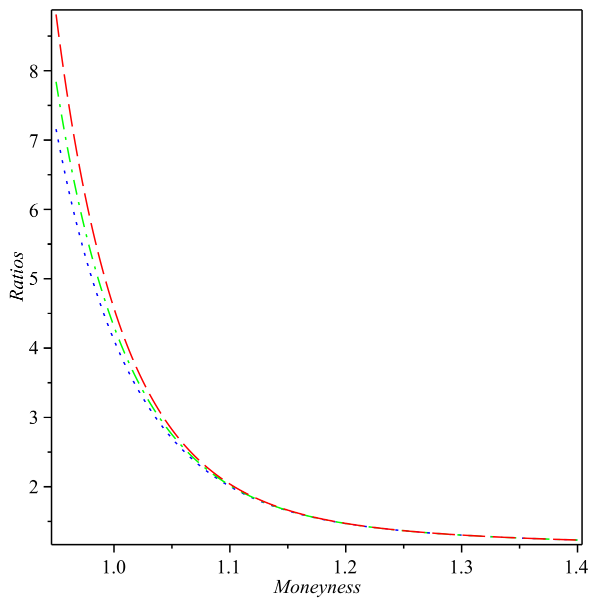
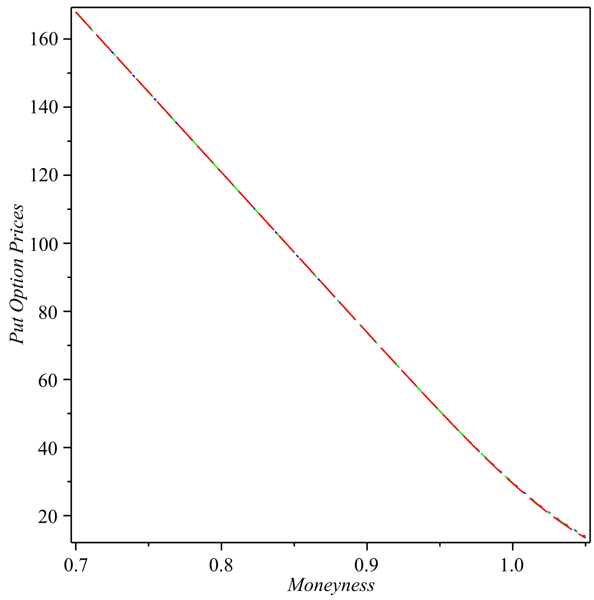


(b)

Figure 6.104: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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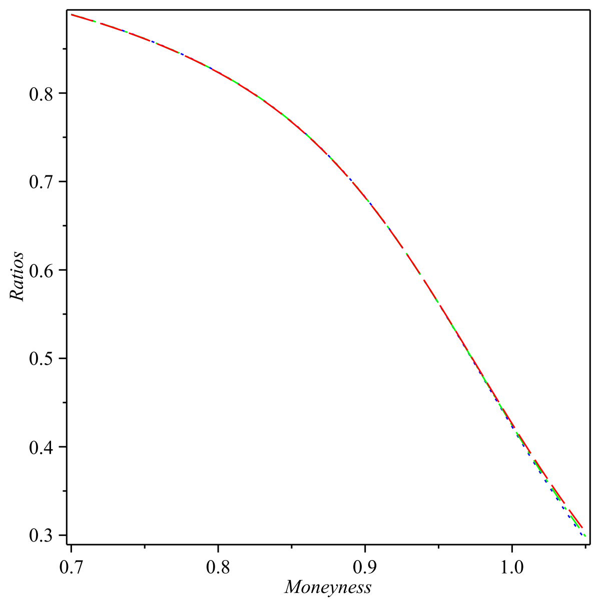
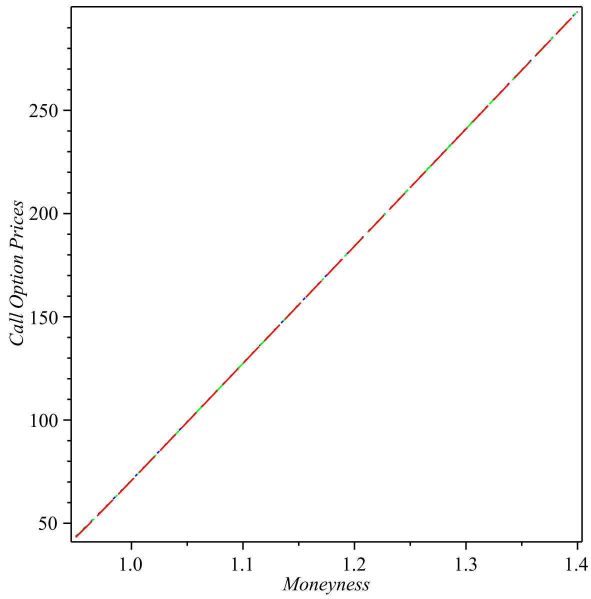


(b)

Figure 6.105: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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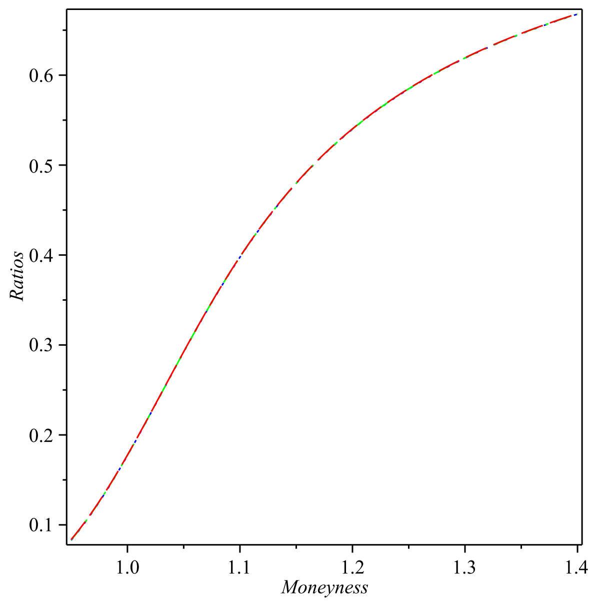
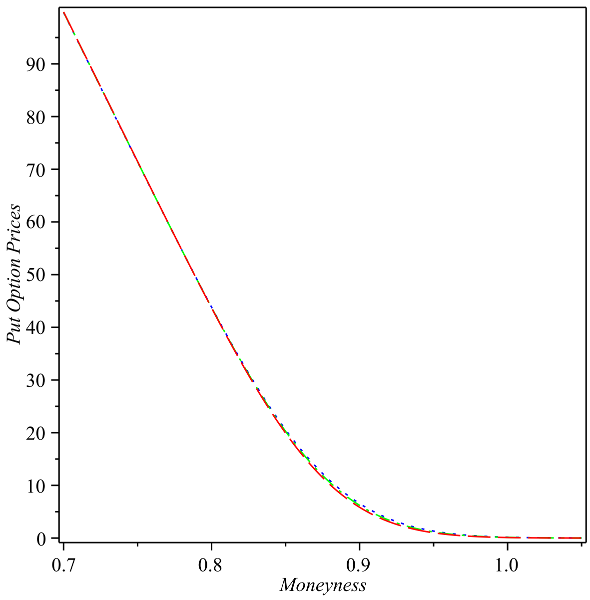


(b)

Figure 6.106: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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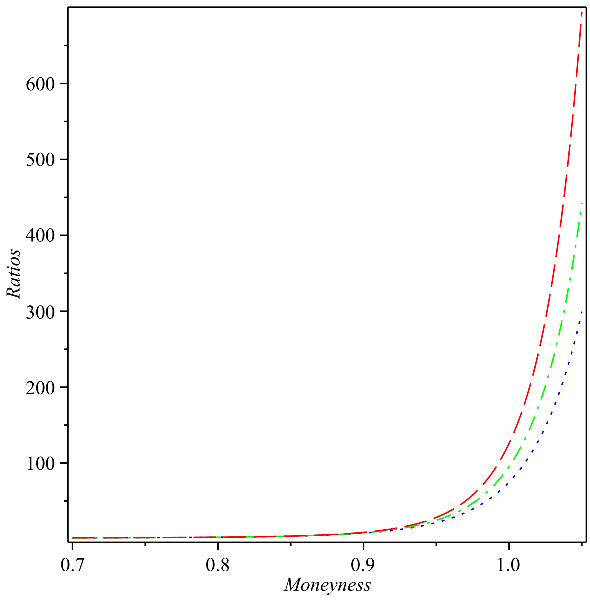
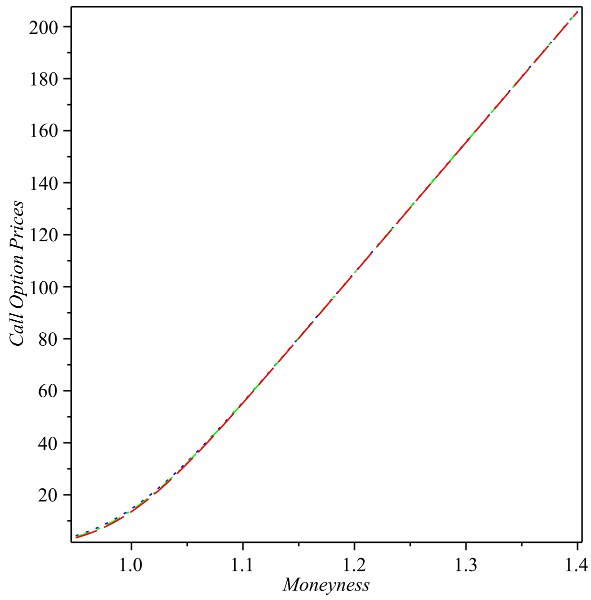


(b)

Figure 6.107: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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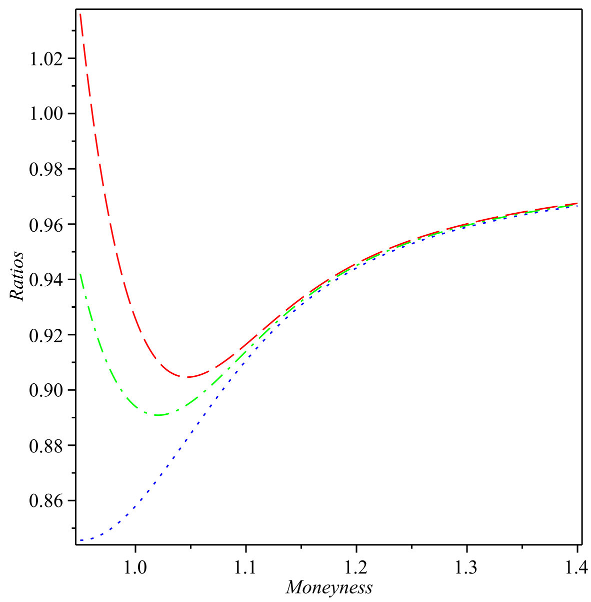
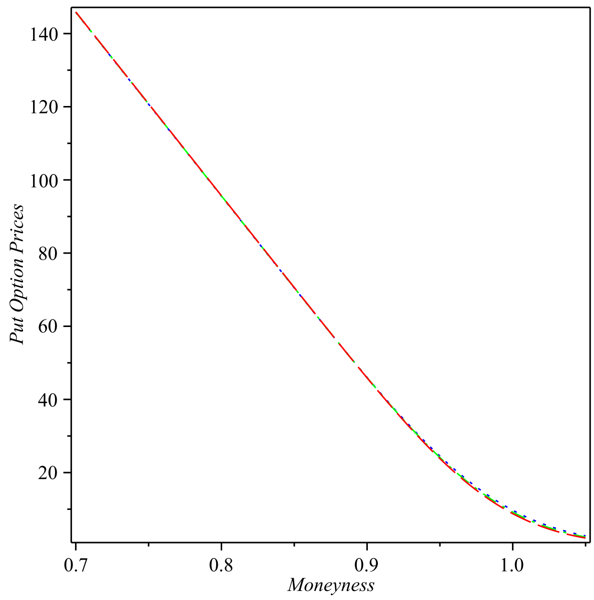


(b)

Figure 6.108: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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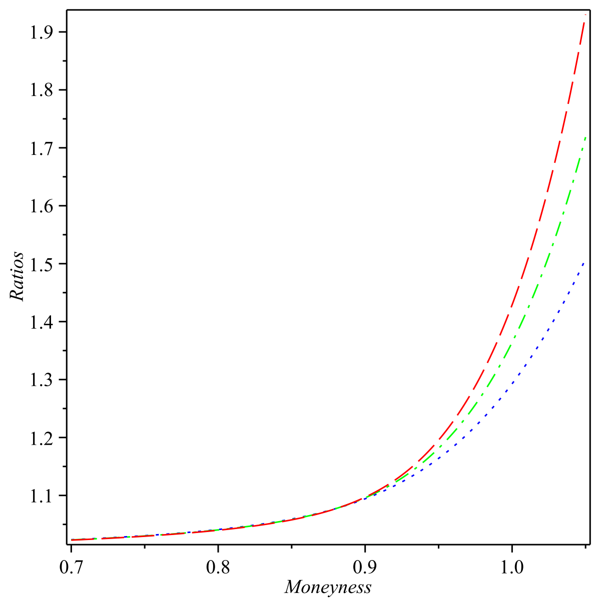
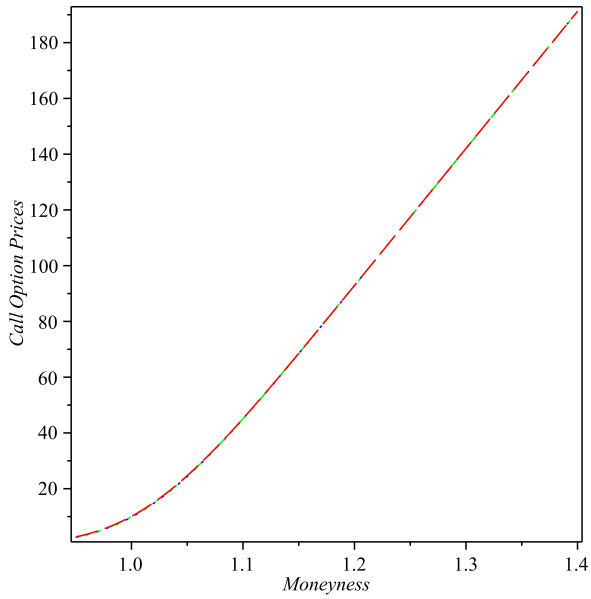


(b)

Figure 6.109: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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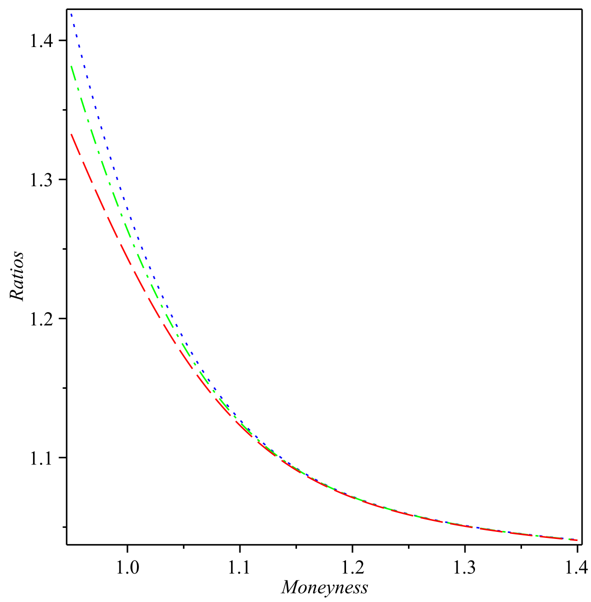
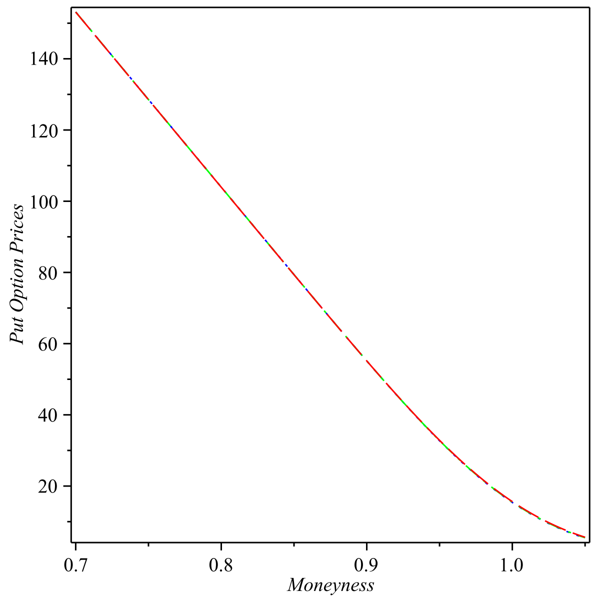


(b)

Figure 6.110: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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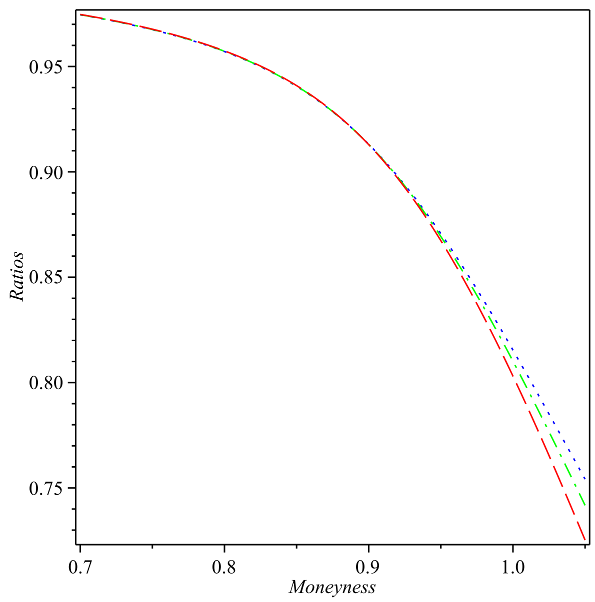
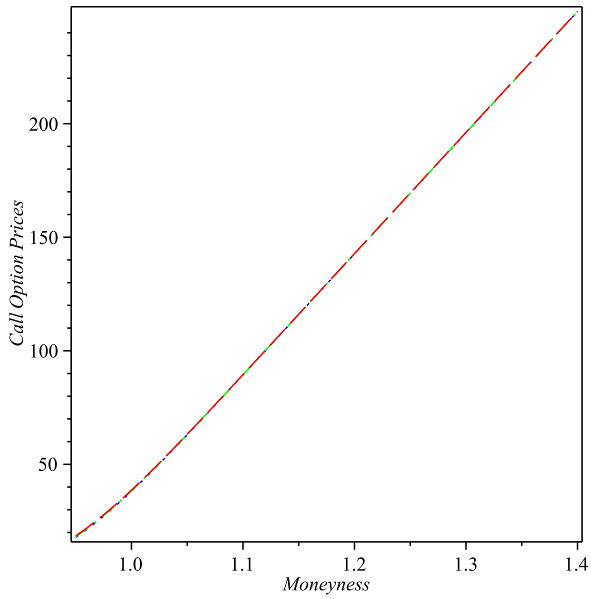


(b)

Figure 6.111: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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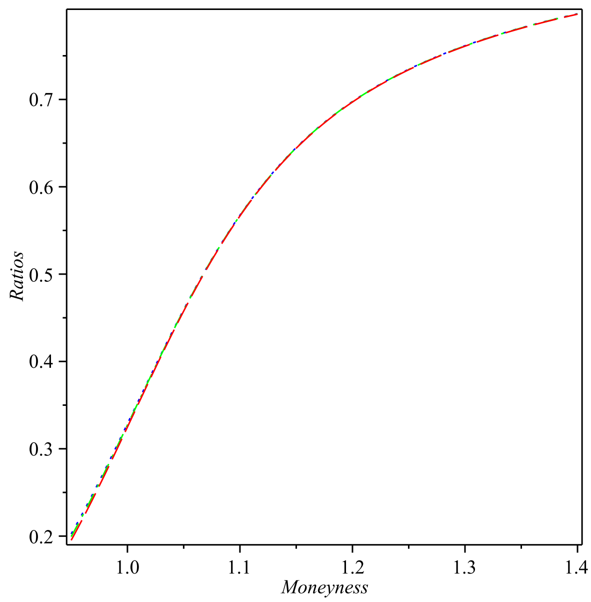
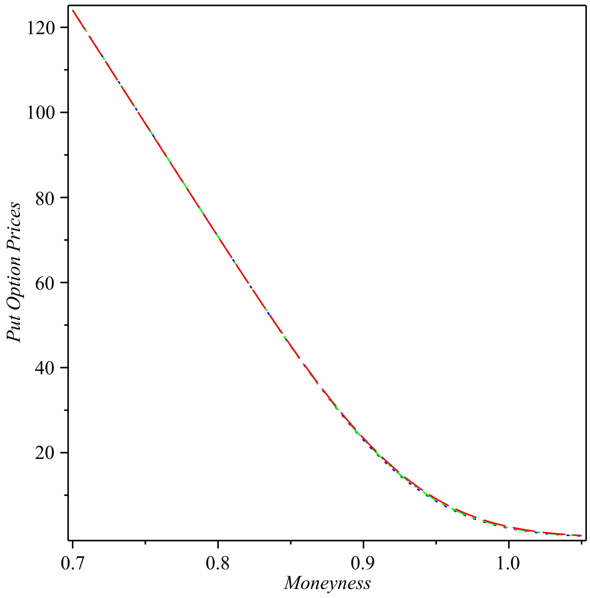


(b)

Figure 6.112: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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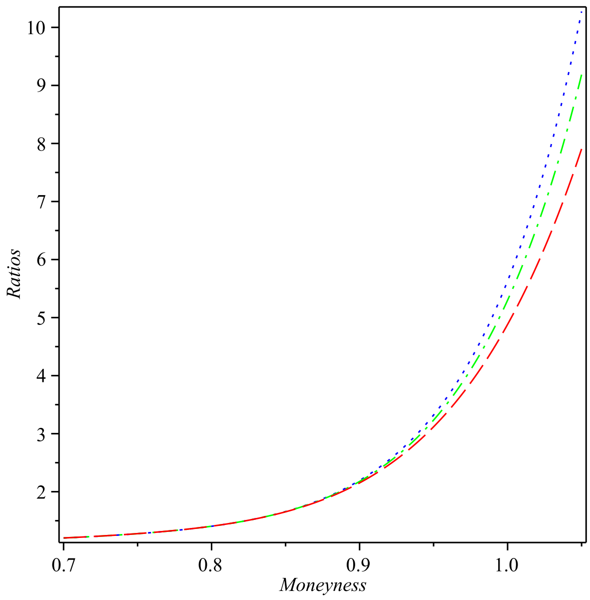
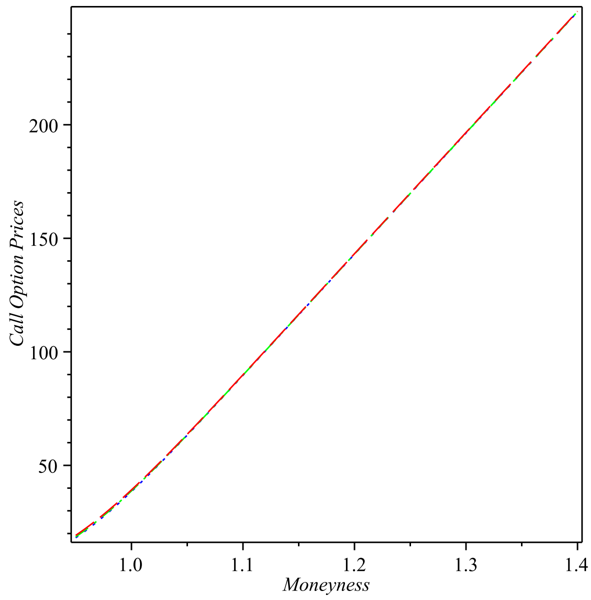


(b)

Figure 6.113: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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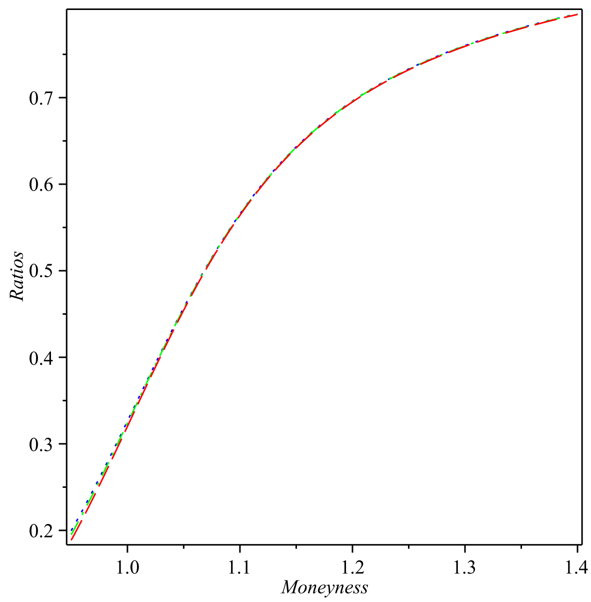
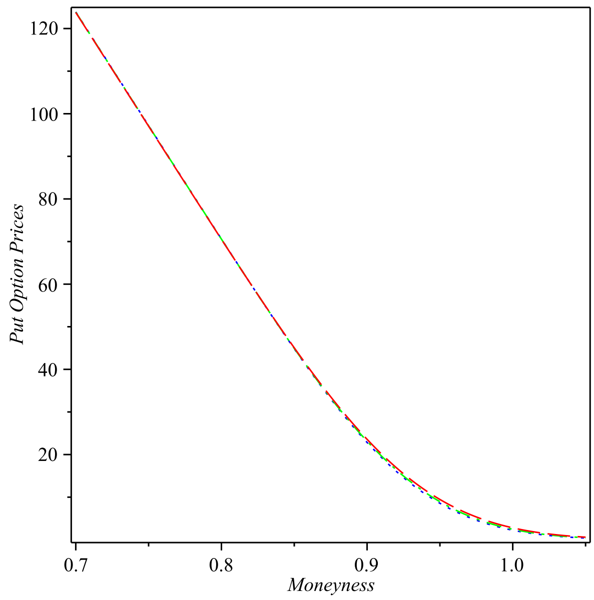


(b)

Figure 6.114: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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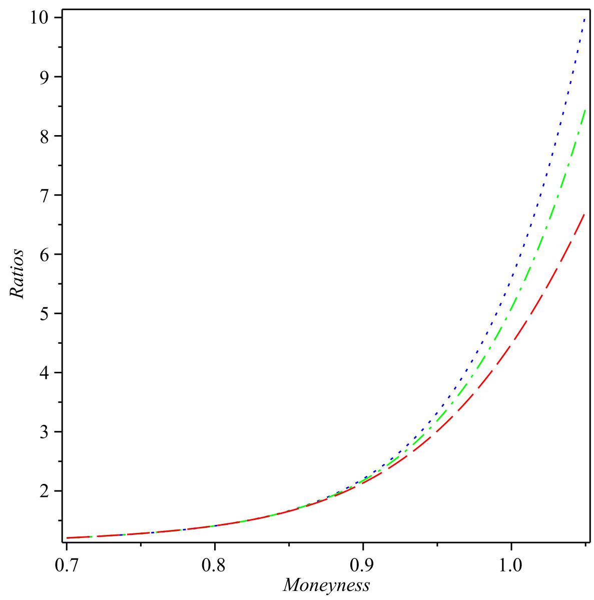
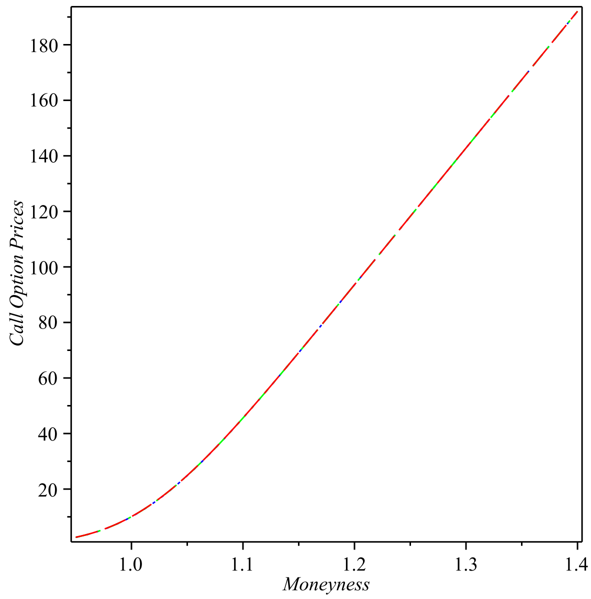


(b)

Figure 6.115: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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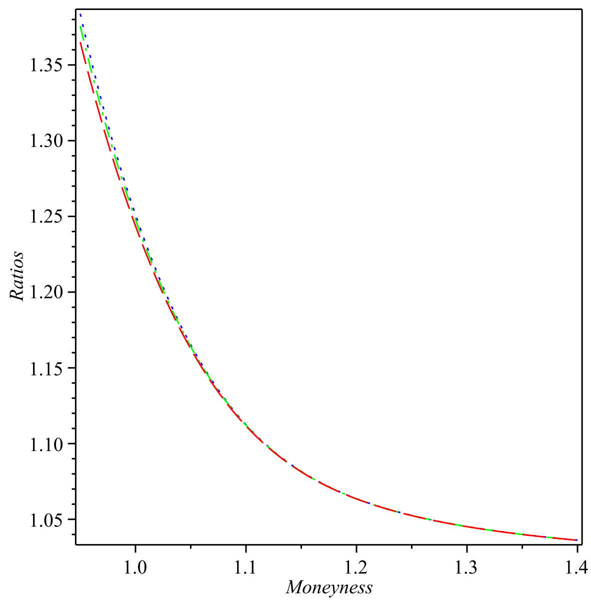
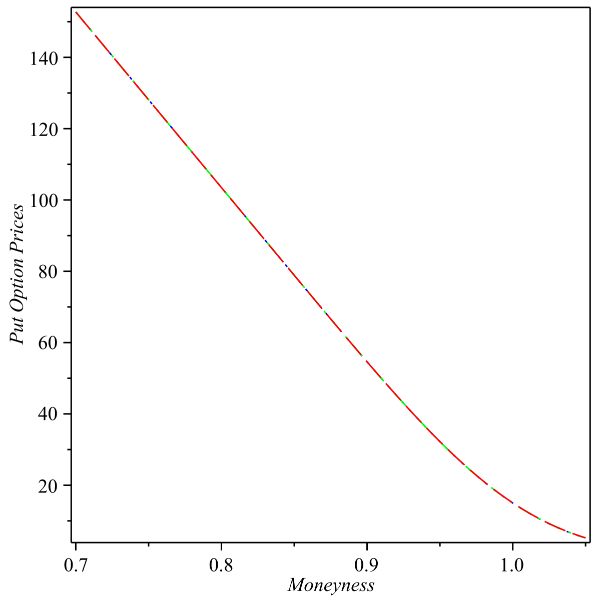


(b)

Figure 6.116: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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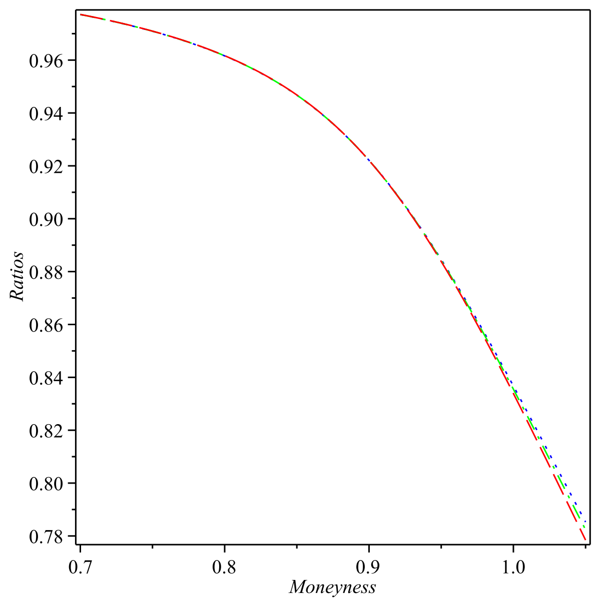
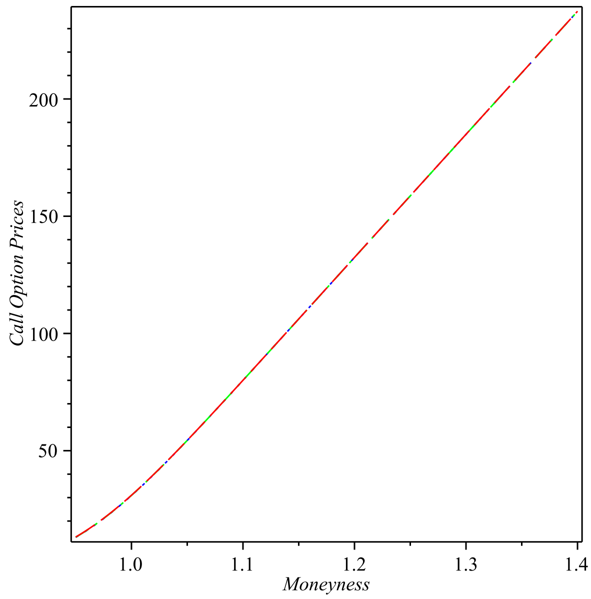


(b)

Figure 6.117: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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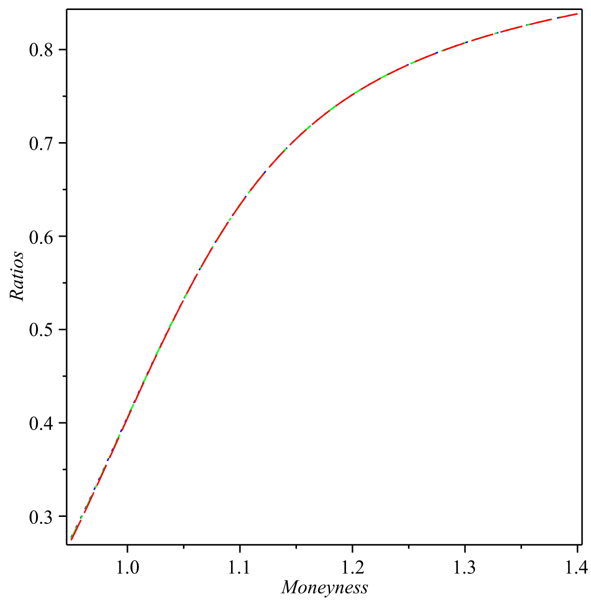
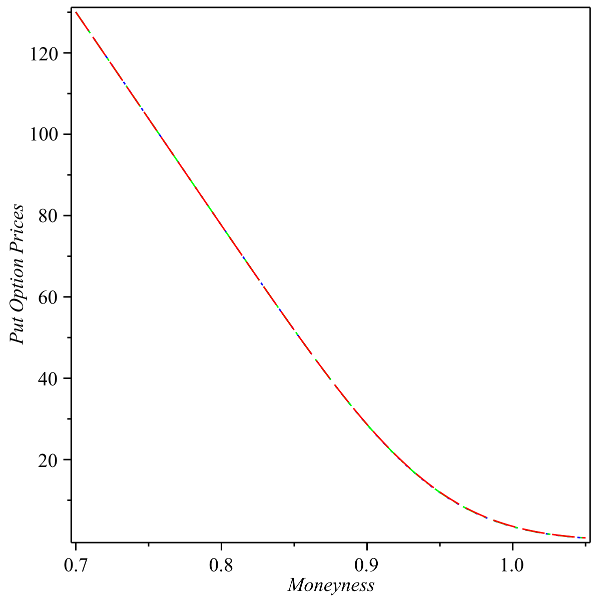


(b)

Figure 6.118: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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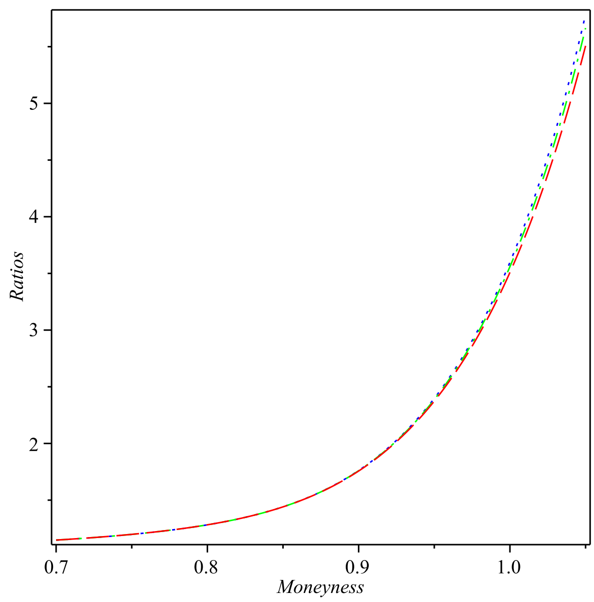
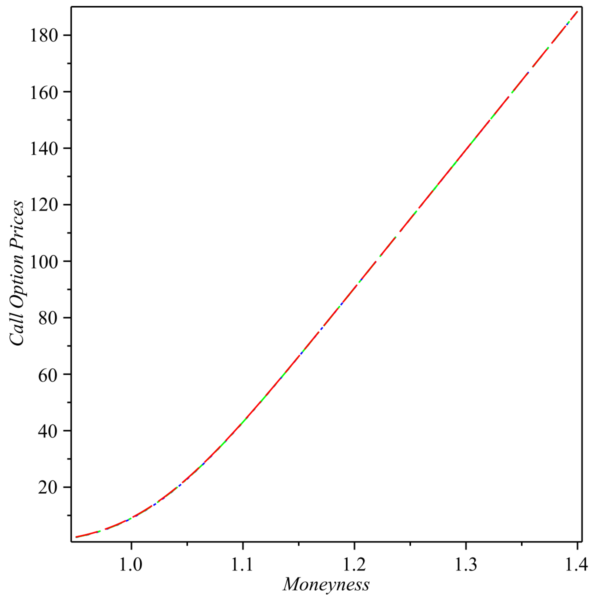


(b)

Figure 6.119: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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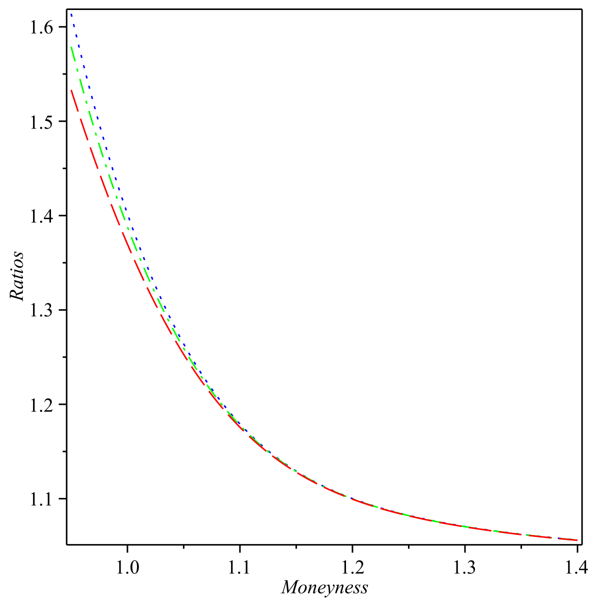
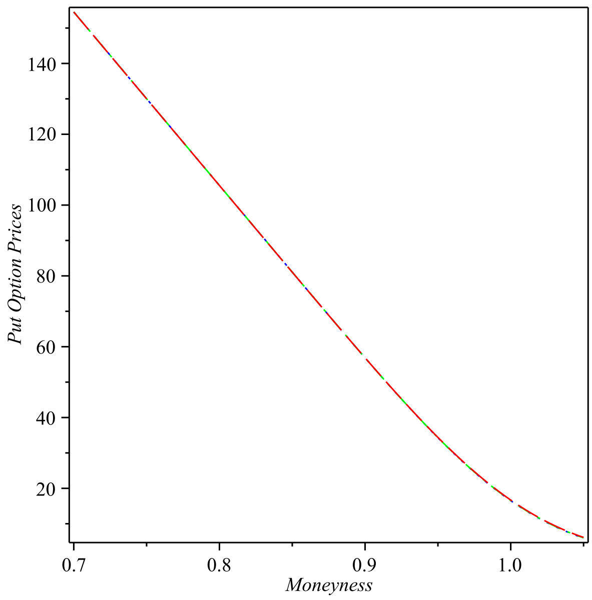


(b)

Figure 6.120: Sensitivity of call option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price, (b) The ratio of call option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).



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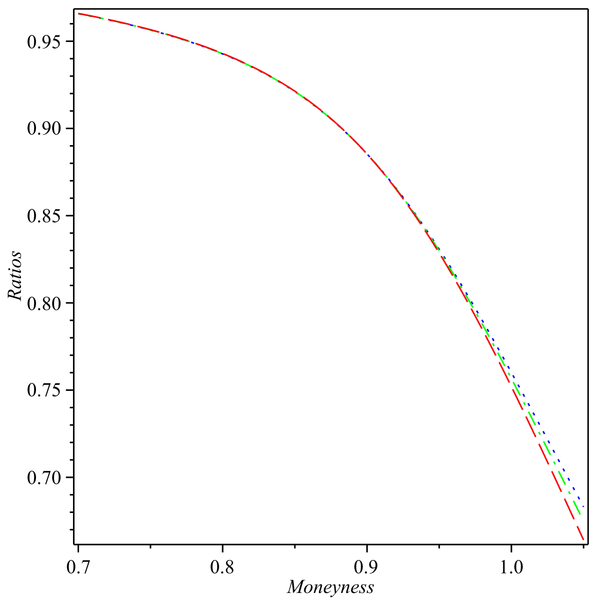


(b)

Figure 6.121: Sensitivity of put option prices to the market price of risk volatility σ_{κ_i} for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Put option price, (b) The ratio of put option price determined by the constant Black-Scholes-Merton model over the option price obtained from our model. In the figure, $\sigma_{\kappa_i} = 0.20$ (dotted line), $\sigma_{\kappa_i} = 0.50$ (dash-dotted line), $\sigma_{\kappa_i} = 0.90$ (dashed line).

As we can expect, the results above show that the market price of risk mostly has a slight impact on the option prices. However in some situations, especially for the out-of-the-money option and deep-out-of-the-money option, the sensitivity of this parameter is at a very significant level, which implies that the change of market price of risk volatility may make a very significant impact on the out-of-the-money option prices. As expected, the option may be overpriced or underpriced owing to the effect of the other stochastic parameters.

6.3 Model Characteristics

For more than three decades, after Black and Sholes brought the outstanding option pricing model to the financial world, many researchers have investigated and modified the option pricing model with the aim of developing a more accurate model.

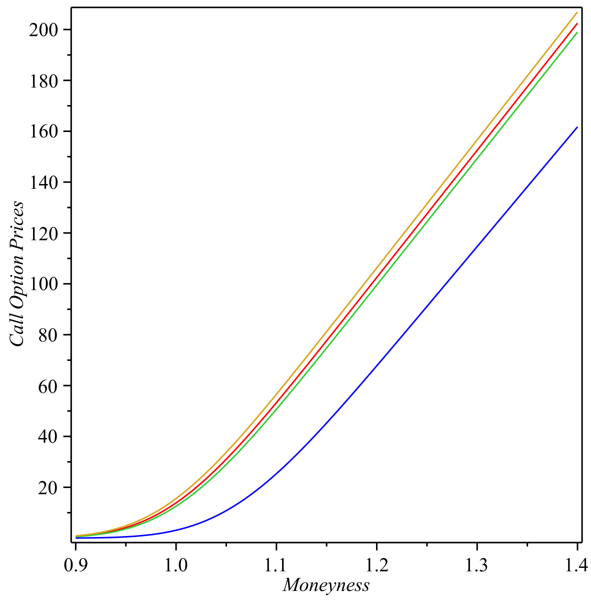
In this thesis, we have taken into account another parameter, the stochastic earning yields, in the option pricing model in order to price the option more realistically. In this section, we will investigate the characteristics of our extended model by comparing it to the other three popular models: the classical Black-Scholes model, the constant dividend Black-Scholes-Merton model and the stochastic dividend yield model.

In this work, we simulate and compare the call option prices and put option prices obtained by different models. The parameters required for the models are set at the values as shown in the Table 6.11. We simulate the models characteristics under various situations of the market friction by setting different friction Parameter values for θ_δ and θ_ξ. The results of simulations are shown in Figures 6.122-6.130.

Table 6.11: Parameter values used in the simulation of models characteristics

$K = 500$	$\delta(0) = 0.05$	$\xi(0) = 0.05$	$\kappa_1(0) = 0.20$
$r = 0.05$	$\mu_\delta = 0.05$	$\mu_\xi = 0.05$	$\kappa_2(0) = 0.20$
$T = 0.1$	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\kappa_3(0) = 0.20$
$\sigma_S = 0.05$	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_1} = 0.20$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 3} = 0.05$	$\mu_{\kappa_2} = 0.20$
			$\mu_{\kappa_3} = 0.20$
			$\theta_{\kappa 1} = 0.45$
			$\theta_{\kappa 2} = 0.45$
			$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 2} = 0.10$
			$\sigma_{\kappa 3} = 0.10$

the classical Black-Scholes model, the constant dividend Black-Scholes-Merton model and the stochastic dividend yield model



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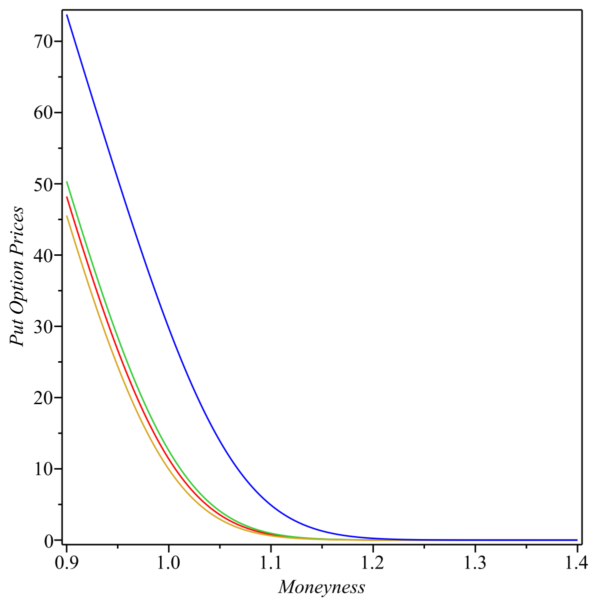
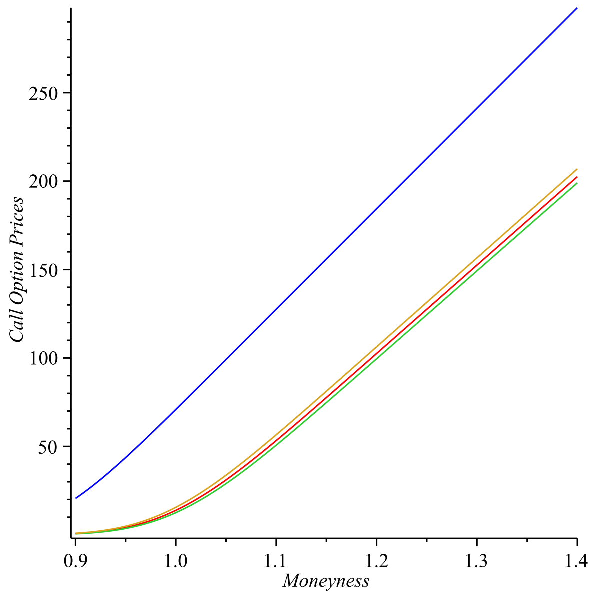


(b)

Figure 6.122: Option price comparison for $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).



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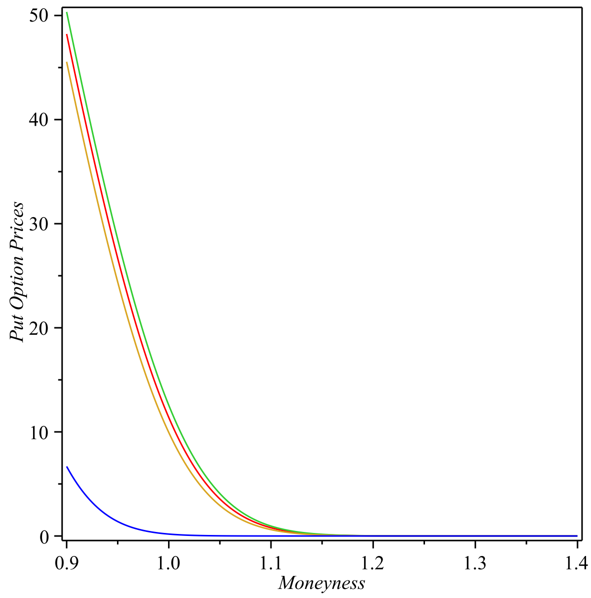
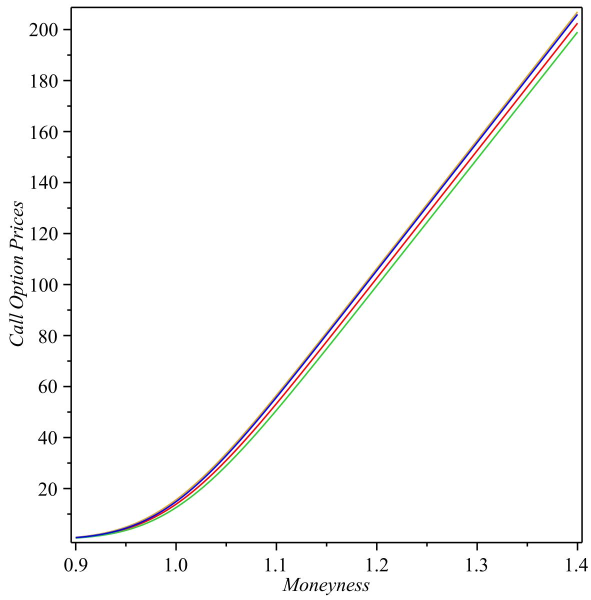


(b)

Figure 6.123: Option price comparison for $\theta_\delta = 0.35$ and $\theta_\xi = 0.55$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).



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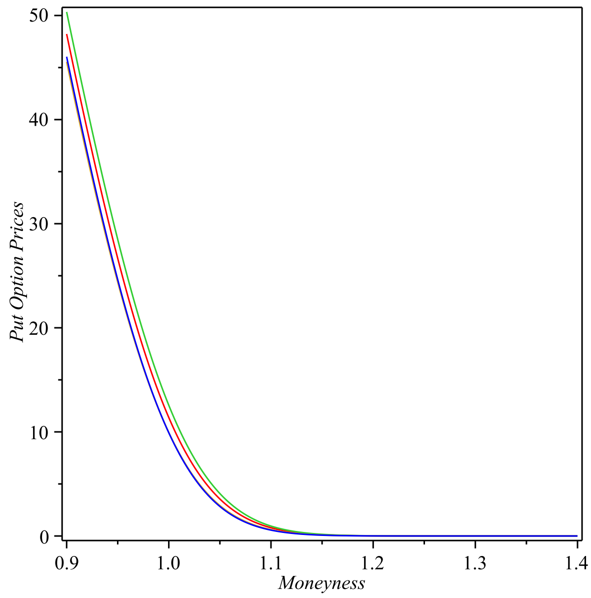
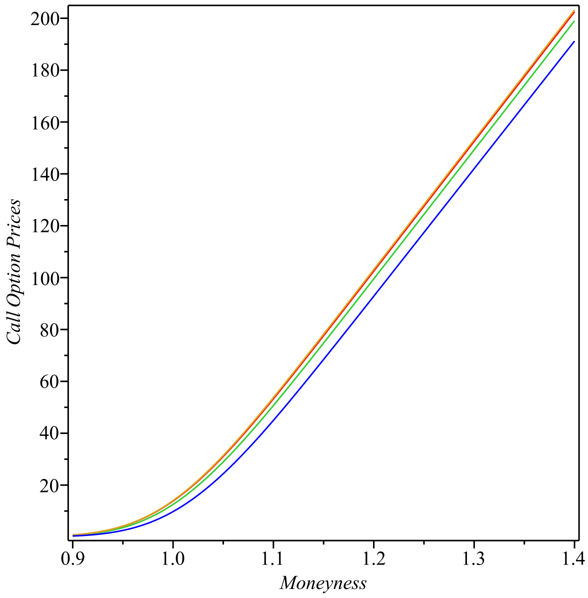
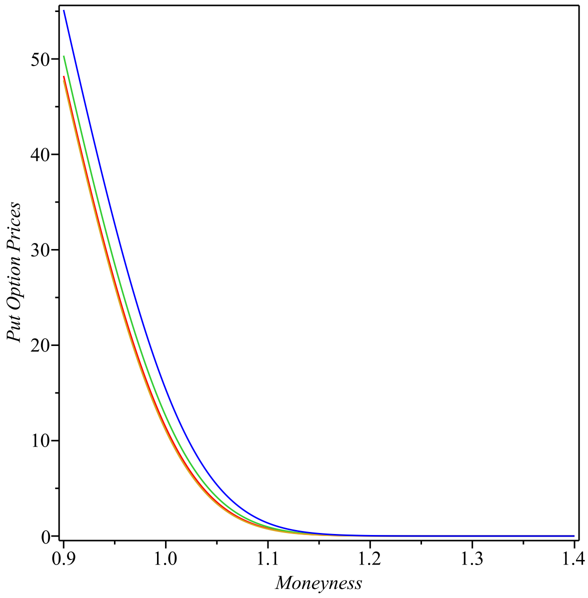


(b)

Figure 6.124: Option price comparison for $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).

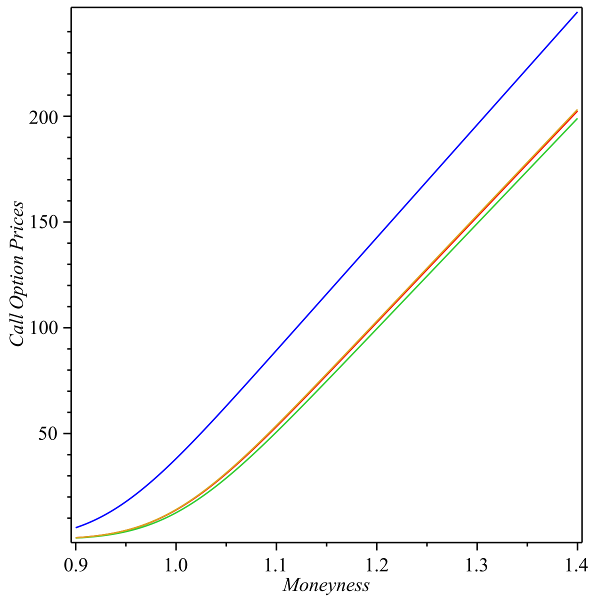


(a)

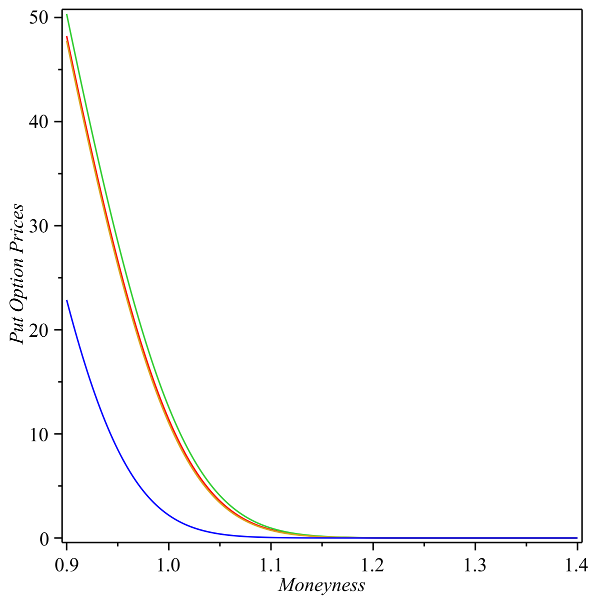


(b)

Figure 6.125: Option price comparison for $\theta_\delta = 0.65$ and $\theta_\xi = 0.25$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).

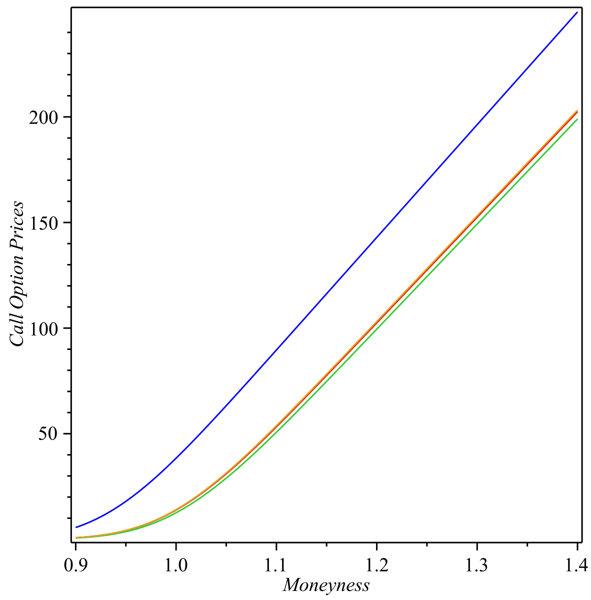


(a)



(b)

Figure 6.126: Option price comparison for $\theta_\delta = 0.65$ and $\theta_\xi = 0.55$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).



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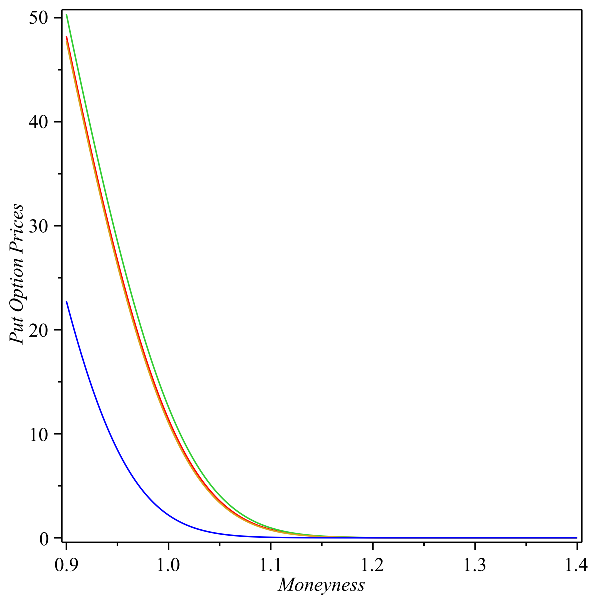
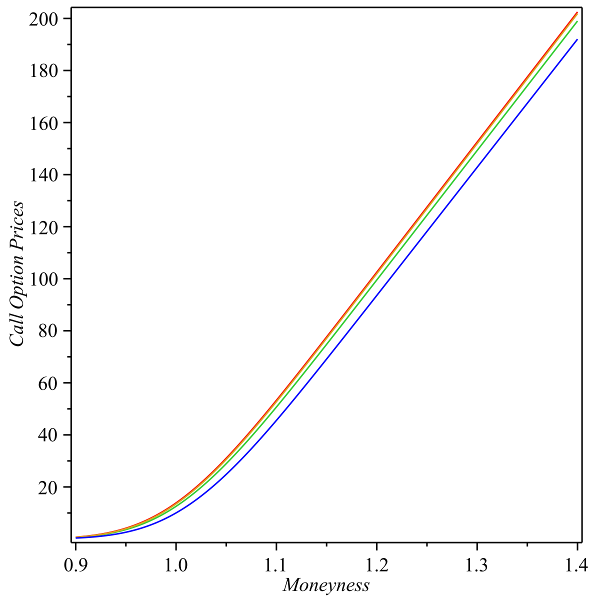


(b)

Figure 6.127: Option price comparison for $\theta_\delta = 0.65$ and $\theta_\xi = 0.85$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).



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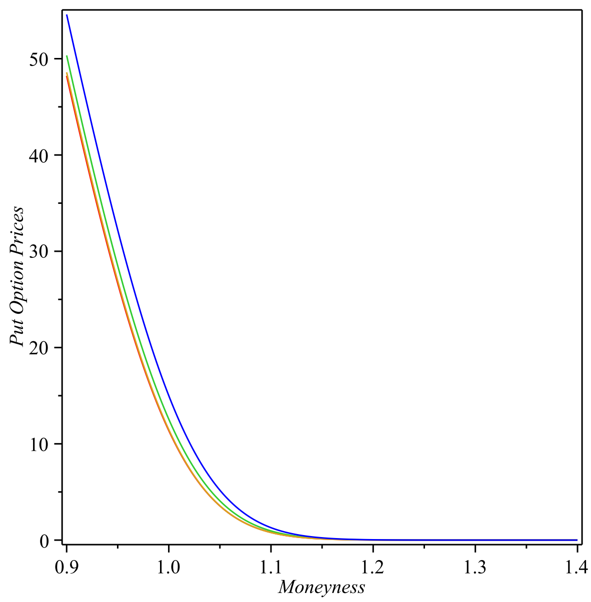
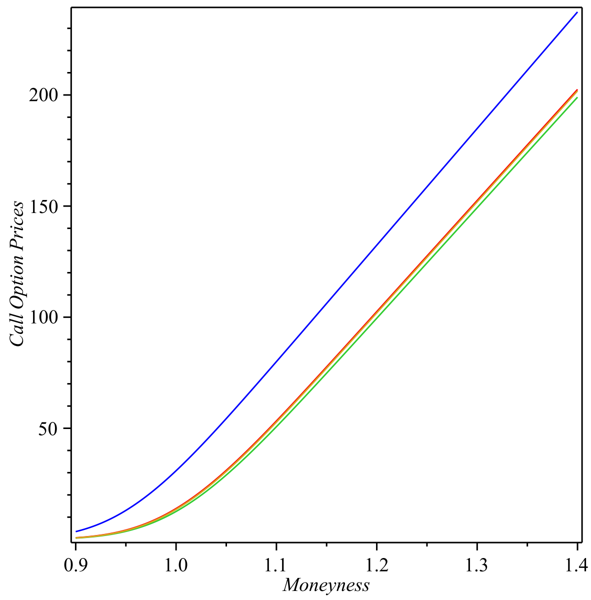


(b)

Figure 6.128: Option price comparison for $\theta_\delta = 0.95$ and $\theta_\xi = 0.25$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).



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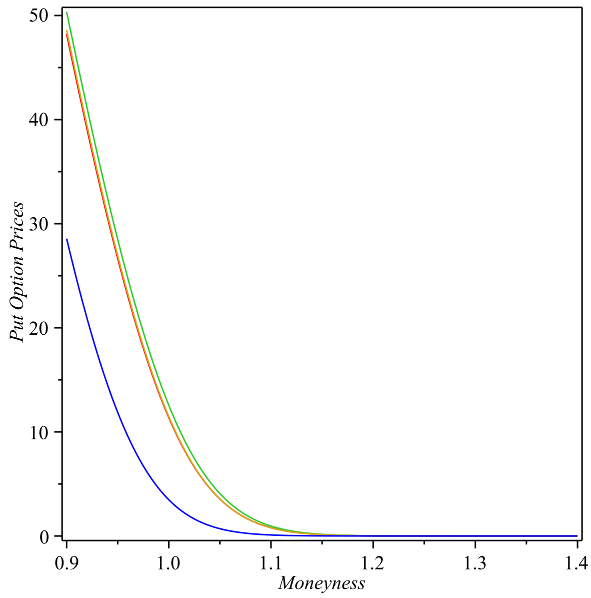
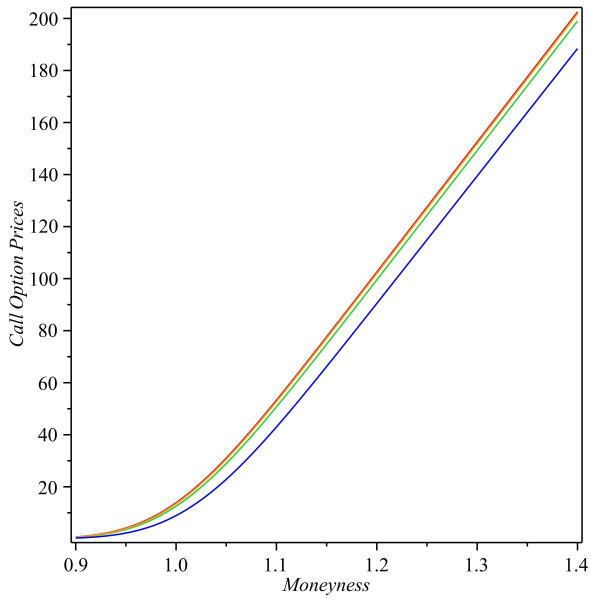


(b)

Figure 6.129: Option price comparison for $\theta_\delta = 0.95$ and $\theta_\xi = 0.55$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).



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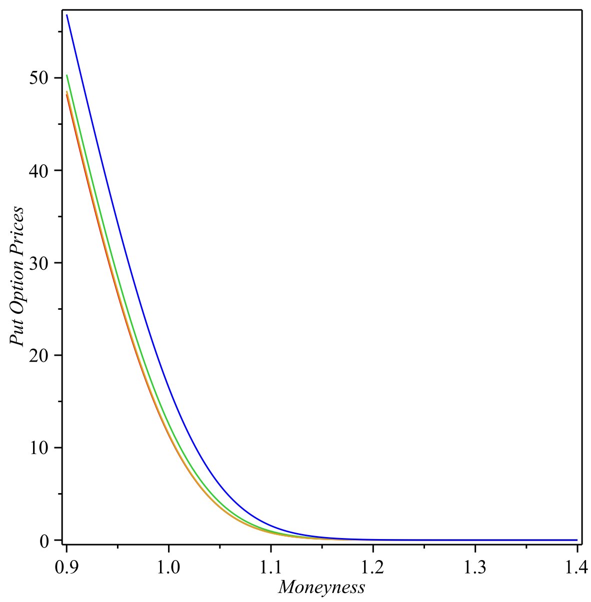

(b)

Figure 6.130: Option price comparison for $\theta_\delta = 0.95$ and $\theta_\xi = 0.85$. (a) Call option price comparison, (b) Put option price comparison. In the figure, the Black-Scholoes model (red line), the constant dividend Black-Scholes-Merton model (green line), the stochastic dividend yield model (yellow line), the stochastic earning yield model (blue line).

The simulations have demonstrated the characteristics of our option pricing model taking account into stochastic dividend yield, stochastic market price of risk and stochastic earning yield. Our model has the same pattern of the other three models for both call and put options; however, our model performs a strong stochastic character. Since the other three simulated models all perform the option price value at the same range, we can investigate the price characteristic of our model by comparing it to the other models under different situations. For example, under the market situation of $\theta_\delta = 0.35$ and $\theta_\xi = 0.25$, the call option prices are overvalued while the put option prices are undervalued. On the contrary, if the situation of the market is altered: θ_δ is changed to 0.35 and θ_ξ is changed to 0.55, the call option price is undervalued and the put option price is overvalued. However, for some circumstances such as $\theta_\delta = 0.35$ and $\theta_\xi = 0.85$, our model predicts the option prices at the same range of the other three models. In summary, the option price may be overpriced or underpriced or valued at the same price compared to the other models depending on the uncertain situation of the variety of market frictions in each source.

6.4 Implied Volatility

The implied volatility for a derivative security such as an option is the volatility of the option price which is implied by the financial market which can be derived by the pricing model, for instance, a Black-Scholes model.

In this section, we observe the implied volatility pattern of the option prices of our model by using the Black-Sholes-Merton model with standard deterministic dividend yield as the model to derive the implied volatility. We simulate the call option price by computing option prices in our model in such a different maturity by setting the parameters as shown in the table 6.12. And we mimic our option prices data as the market data of call option prices. The results are demonstrated in the following figures. Figure 6.131 exhibits the implied volatilities when $T = 0.1$ and figure 6.132 exhibits the implied volatilities when $T = 0.3$.

Table 6.12: Parameter values used for the simulation of models characteristics

$K = 500$	$\theta_\delta = 0.35$	$\theta_\xi = 0.25$	$\kappa_1(0) = 0.20$
$r = 0.05$	$\delta(0) = 0.05$	$\xi(0) = 0.05$	$\kappa_2(0) = 0.20$
$\sigma_S = 0.05$	$\mu_\delta = 0.05$	$\mu_\xi = 0.05$	$\kappa_3(0) = 0.20$
	$\sigma_{\delta 1} = 0.05$	$\sigma_{\xi 1} = 0.05$	$\mu_{\kappa_1} = 0.20$
	$\sigma_{\delta 2} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_2} = 0.20$
	$\sigma_{\delta 3} = 0.05$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_3} = 0.20$
			$\theta_{\kappa 1} = 0.45$
			$\theta_{\kappa 2} = 0.45$
			$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.10$
			$\sigma_{\kappa 2} = 0.10$
			$\sigma_{\kappa 3} = 0.10$

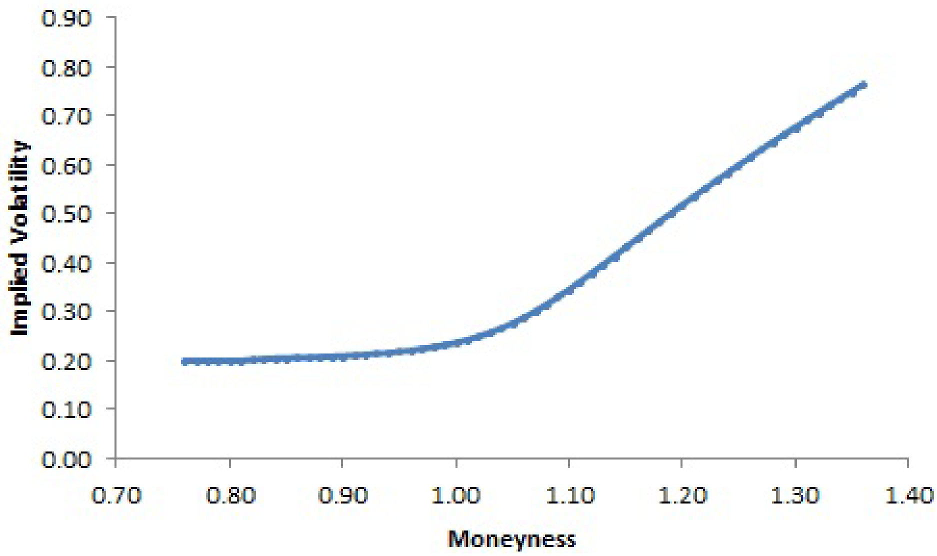

Figure 6.131: Implied volatilities pattern for T=0.1

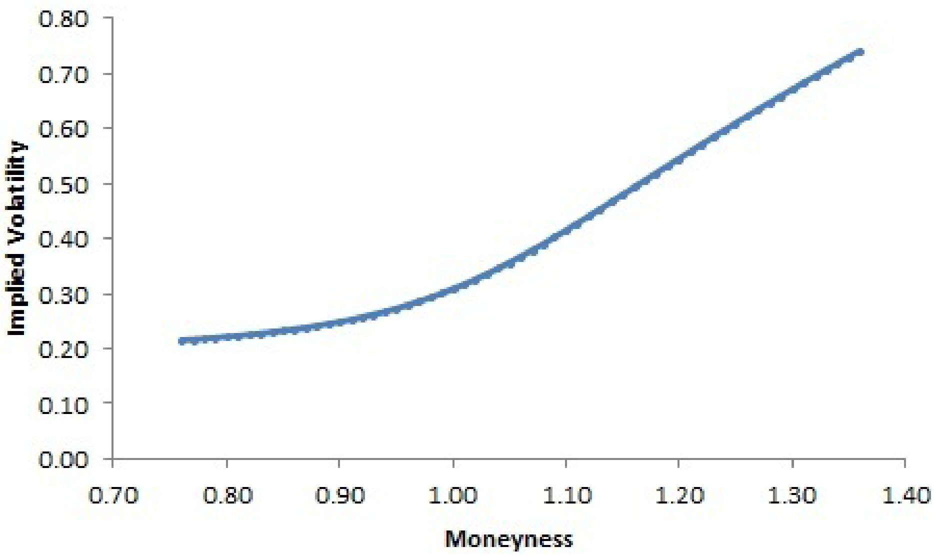

Figure 6.132: Implied volatilities pattern for T=0.3

The call option prices of our extended model yields implied volatility patterns similar to the call option prices in the real-world markets. This property is required as explanation to demonstrate a model as a suitable and applicable option pricing model. Our model, as an extension of the existing model taking into account the stochastic dividend yield, stochastic market price of risk and especially the stochastic earning yield, has given very close properties of the real world option prices.

6.5 Concluding Remarks

This chapter investigates the performance of our extended option pricing model taking into account the stochastic dividend yield, stochastic market price of risk and stochastic earning yield. This work is important as the main question of option pricing model research is to investigate whether the new model is applicable or not and whether it can provide the option price that is close to the real world data or not.

The sensitivity analysis is examined on the parameters in our model. The sensitivity test provides useful information of our model. The friction coefficient in both dividend yield and earning yield has played the most significant role for the option pricing, since they are the parameters related to the real-world uncertainty in any economic situation. Under different friction of the market, the stock prices vary and the option prices also vary accordingly. Moreover under certain combination of frictions from different source of the market, reflecting different situation in the real world market, the option prices can be impacted dramatically. The market price of risk friction coefficient makes an impact to the option prices at a low level; however, they still can impact the option price and even with the outstanding level of sensitivity in some situations. Maturity is the parameter to pay attention since the results show significant level of sensitivity. The changing of stock volatility value almost has no impact to the in-the-money options; nevertheless, still, it makes a vast impact to the out-of-the-money options. Dividend yield volatility and earning yield volatility play an important role to the option price since they yield various range of option prices under different situation of market frictions. The volatility of market price of risk shows low sensitivity to the option prices for some cases, but has high level impacts in some other circumstances. The results of the model testings have demonstrated the real-world property of uncertainty. The option may be undervalued and overvalued at any time and at any situation.

In this chapter, we also explore the characteristic of our option pricing model. The results also capture the outstanding property of the real-world market namely the uncertainty of the financial world. Whereas, the model still accomplishes the common property of the option price as shown in the implied volatility.

In conclusion, our model has performed a very powerful stochastic properties, reflecting the uncertainty of the real-world market. The option price model taking into account the stochastic earning yield, stochastic dividend yield and stochastic market price of risk is a serious candidate to explain the option price in the current world market.

Chapter 7

Application of Option Pricing Model with Stochastic Earning Yield

7.1 General

In financial world, stock traders always pay their attention on stock movement. Some might guess the stock movement by analyzing the company profile and reading news; some others might apply some mathematical techniques to judge their disciplined process. The uncertainty of the stock market is the key factor for some investors to seek for certain security of their investments in order to accomplish their highest goal.

Option is one of the derivative security which plays an interesting role in the financial market. Some traders purchase some options for playing the next move of their trades to take advantage of the underlying movement.

The traders should understand the factors which affect the option prices. For decades, it has been taught that the option prices are impacted by six factors: stock price, dividend yield, time to maturity, volatility, strike price and stock price. Some mathematical researchers have brought this idea to establish the option pricing models which apply these parameters to estimate the "fair market value" of the option. The very well known Black-Scholes option pricing model is an obvious example and is widely used by some of the investors. Consequently, many investors take advantage of the moves of the market by considering the fair market value.

In our research, we contribute to the development of the new extended option pricing model by considering the stochastic earning yield parameter. By this, we obtain the theoretical option pricing formula for determining the fair market value price by using more realistic data since we input one more parameter to consider the option price which is "earning yield" or a reciprocal of the "P/E ratio" as it can be seen in the daily newspaper in a stock data section. In this section we will perceive the application of our extended model.

7.2 Comparison of Option Pricing Model

In this chapter, we apply the real-world data to observe the consistency of the option pricing model taking into account stochastic dividend yield, stochastic market price of risk and stochastic earning yield. There are four models to investigate: Black-Scholes model, Black-Scholes-Merton model with Constant Dividend Yield, Stochastic dividend yield model and Stochastic earning yield model. To compare the models, we use the sum of squared errors as a measure, that is the sum of the squared errors between the option prices generated by the option pricing model and the option prices acquired from the real-world financial markets. Theoretically, the smaller the squared error is, the higher consistency the option pricing model has with the data.

The sum of squared errors (also known as residual sum of squared) is a measure of the difference between the real data and the data obtained from the estimation model. A small value of the sum of squared errors designate a good fit of the model to the real data. The sum of squared errors, to be examined for each model, is defined by

$$\text{Errors} = \sum_{i=1}^{n} (y_i - F(K_i, t))^2 \tag{7.1}$$

where y_i is the i^{th} real-market data and $F(K_i, t)$ is the option prices obtained from the option pricing models for either call or put option. The real-market data, y_i, is the closing price of call or put option which can be obtained from the market. In this thesis, the data is brought out from the well known source: http://finance.yahoo.com as of Friday, March 02, 2012. And the model prices, for call and put option, are given by each corresponding model. The sum of squared errors among the investigated models are then compared.

Our goal in this section is to compare the option pricing models, the real-market data required. Since our model is applied to the European option, we choose to investigate the price from the stock indexes at different maturity for each stock index. We simulate four stock indexes: 1/100 Dow Jones Industrial Average (ˆDJX), S&P 500 (ˆGSPC), Russell 2000 (ˆRUT) and NASDAQ-100 (ˆNDX) as they provide European option prices. We gain some other significant information, such as P/E ratio and dividend yield, from these stock indexes from the Wall Street journal website: http://online.wsj.com. Moreover, by exploring the information from the Wall Street source, we can have the mean of the dividend yield and the mean of the P/E ratio. In conclusion, we can calculate the input values directly for the parameters of the models, including current stock price ($S(0)$), maturity (T), risk-free interest rate (r), current dividend yield ($\delta(0)$), average dividend yield (μ_δ), current earning yield ($\xi(0)$) and average earning yield (μ_ξ). For the other parameters, we apply the bisection method to find the best unknown parameters for our option pricing model by minimizing the sum of squared errors. Since we aim for the smallest value of the sum of squared error as well as the large number of decimal digits significantly affects the sum of squared error, please note that the numerical calculations shown in the table in this chapter are in the form of a large number of decimal digits.

7.2.1 1/100 Dow Jones Industrial Average (ˆDJX)

The stock index 1/100 Dow Jones Industrial Average (ˆDJX) is an index which shows the average performance of the trade of the 30 major American companies in the stock market [81]. In order to do the simulation by the models, we first find the σ_S for the least sum of squared errors of the Black-Scholes model for call options. Then we use the σ_S for the other models to find the sum of squared errors in each model. The parameters used are as shown in the table 7.1. The data set is acquired on the date of Friday, March 02, 2012 and the options expire on March 16^{th}, 2012.

Table 7.1: Parameters values used in the simulation of the 1/100 Dow Jones Industrial Average option prices

$S = 129.78$	$\theta_\delta = 0.15$	$\theta_\xi = 0.36$	$\kappa_1(0) = 0.20$
$r = 0.053$	$\delta(0) = 0.025$	$\xi(0) = 0.068870$	$\kappa_2(0) = 0.20$
$T = 14/365$	$\mu_\delta = 0.0237$	$\mu_\xi = 0.067159$	$\kappa_3(0) = 0.20$
$\sigma_S = 0.10$	$\sigma_{\delta 1} = 0.01$	$\sigma_{\xi 1} = 0.05$	$\mu_{\kappa_1} = 0.20$
	$\sigma_{\delta 2} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_2} = 0.20$
	$\sigma_{\delta 3} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_3} = 0.20$
			$\theta_{\kappa 1} = 0.45$
			$\theta_{\kappa 2} = 0.45$
			$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.40$
			$\sigma_{\kappa 2} = 0.40$
			$\sigma_{\kappa 3} = 0.40$

The simulation results are obtained for all the four call option pricing models, and are shown in the table 7.2 against the corresponding real-market data of Dow Jones Industrial Average Option Prices (ˆDJX).

We plot the graphs of the results of table 7.2 for all the four call option pricing models as shown in figure 7.1. It is obvious that the call option prices of the real market is fluctuated. The call option prices obtained from the three models, Black-Scholes model (BSC), Black-Scholes-Merton model with constant dividend yield (CDC) and stochastic dividend yield (SDC), are all in the same range; while the option prices obtained from the model with stochastic earning yield (SEC) is in the middle of the real-market data (ˆDJX) which shows that our model gives the best fitting.

Table 7.2: Comparison of real-market call option prices of DJX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	DJX	BSC	CDC	SDC	SEC
95	26.54	34.97292712	34.84854022	34.89491572	30.82105882
96	23.45	33.97495792	33.85057102	33.89694652	29.82308962
97	24.61	32.97698874	32.85260184	32.89897734	28.82512044
98	21.3	31.97901956	31.85463266	31.90100816	27.82715126
99	29.15	30.98105036	30.85666346	30.90303896	26.82918206
100	19.67	29.98308118	29.85869428	29.90506978	25.83121288
101	28.3	28.985112	28.8607251	28.9071006	24.8332437
102	18.5	27.9871428	27.8627559	27.9091314	23.8352745
103	26.3	26.9891736	26.8647867	26.9111622	22.8373053
104	18.95	25.9912044	25.8668175	25.913193	21.8393361
105	23.8	24.9932352	24.8688483	24.9152238	20.8413669
106	15.9	23.9952661	23.8708792	23.9172547	19.8433978
107	14.65	22.9972969	22.87291	22.9192855	18.8454286
108	14.9	21.9993277	21.8749408	21.9213163	17.8474594
109	15.75	21.0013585	20.8769716	20.9233471	16.8494902
110	13.5	20.0033893	19.8790024	19.9253779	15.851521
111	11.25	19.0054201	18.8810332	18.9274087	14.8535518
112	11.5	18.0074509	17.883064	17.9294395	13.8555826
113	16.25	17.0094817	16.8850948	16.9314703	12.85761341
114	14.12	16.0115125	15.8871256	15.9335011	11.85964422
115	14.59	15.0135434	14.8891565	14.93553199	10.86167531
116	9.1	14.0155742	13.8911873	13.93756279	9.86E+00
117	13	13.017605	12.89321812	12.93959361	8.87E+00
118	12	12.0196358	11.89524911	11.94162459	7.87E+00
119	10.5	11.02166669	10.89728154	10.94365688	6.87E+00
120	10.19	10.02369854	9.89932412	9.94569853	5.88E+00
121	8.45	9.02573875	8.90142553	8.94779482	4.89E+00
122	7.41	8.02783803	7.90380908	7.95015527	3.92E+00
					Continued.

Table 7.2 (Continued)

K	DJX	BSC	CDC	SDC	SEC
123	6.5	7.03026856	6.90731812	6.95357776	2.99E+00
124	5.95	6.03418814	5.91457677	5.96056561	2.15E+00
125	4.78	5.04350372	4.93231445	4.97759331	1.43E+00
126	3.8	4.06865307	3.97471326	4.0184257	8.71E-01
127	3.51	3.13160732	3.06617607	3.10696905	4.78E-01
128	2.55	2.26834723	2.24047181	2.27665038	2.35E-01
129	1.68	1.5233354	1.53386966	1.56383998	1.02E-01
130	1.07	0.934599769	0.973636679	0.996471253	3.92E-02
131	0.57	0.51689883	0.56767548	0.58347853	1.32E-02
132	0.28	0.25484119	0.30163467	0.31147705	3.90E-03
133	0.21	0.1110171	0.14515144	0.15063251	1.01E-03
134	0.14	4.25E-02	6.30E-02	6.57E-02	2.27E-04
135	0.05	1.42E-02	2.45E-02	2.57E-02	4.47E-05

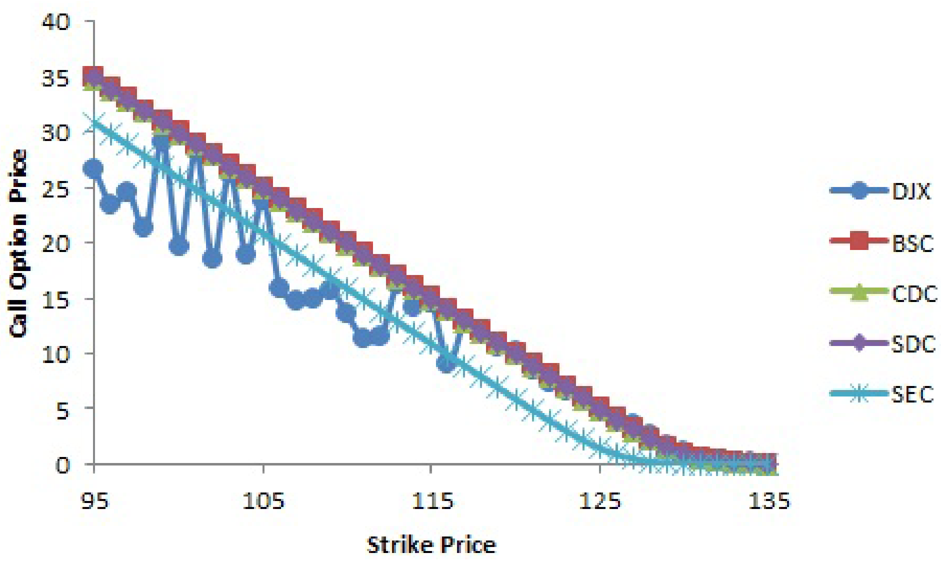

Figure 7.1: Comparison of real-market call option prices of DJX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

In order to compare the model numerically, we compute the sum of squared errors, as defined by (7.2), for the four different models. The results are shown in table 7.3.

Table 7.3: Sums of squared errors of call option prices obtained by four different models for the 1/100 Dow Jones Industrial Average option prices

Model	BSC	CDC	SDC	SEC
Sum of Squared Errors	1005.807276	974.1396656	985.7880812	486.3790972

The result in the table shows that our model, the option pricing model taking into account stochastic dividend yield, stochastic market price of risk and stochastic earning yield, has the smallest sum of squared errors which is less than half of the other models. This indicates that our option pricing model is more effective than the previous option pricing models.

Next, we will explore the put option data by using the same procedure. Table 7.4 shows the simulation results from the four models for the put option prices, together with the real market data for the index of Dow Jones Industrial Average (ˆDJX) as of Friday, March 02, 2012.

Table 7.4: Comparison of real-market put option prices of DJX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	DJX	BSP	CDP	SDP	SEP
95	0.22	0.00E+00	0.00E+00	0.00E+00	0.00E+00
96	1.85	0.00E+00	0.00E+00	0.00E+00	0.00E+00
97	0.27	0.00E+00	0.00E+00	0.00E+00	0.00E+00
98	0.04	0.00E+00	0.00E+00	0.00E+00	0.00E+00
99	0.15	0.00E+00	0.00E+00	0.00E+00	0.00E+00
100	0.33	0.00E+00	0.00E+00	0.00E+00	0.00E+00
101	1.48	0.00E+00	0.00E+00	0.00E+00	0.00E+00
102	1.09	0.00E+00	0.00E+00	0.00E+00	0.00E+00
103	0.05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
104	1.9	0.00E+00	0.00E+00	0.00E+00	0.00E+00
105	0.01	0.00E+00	0.00E+00	0.00E+00	0.00E+00
106	0.12	0.00E+00	0.00E+00	0.00E+00	0.00E+00
107	2.21	0.00E+00	0	0.00E+00	0.00E+00
108	2.91	0.00E+00	0	0.00E+00	0.00E+00
109	0.46	0.00E+00	0	0.00E+00	0.00E+00
110	0.12	0.00E+00	0	0.00E+00	0.00E+00
111	0.3	0.00E+00	0	0.00E+00	0.00E+00
112	0.46	0.00E+00	0	0.00E+00	0.00E+00
113	0.1	0.00E+00	0	0.00E+00	1.00E-08
114	0.11	0.00E+00	0	0.00E+00	2.00E-08
115	0.05	0.00E+00	0	-1.00E-08	2.10E-07
116	0.05	0.00E+00	0	-1.00E-08	2.29E-06
117	0.22	1.00E-08	2.00E-08	1.00E-08	1.94E-05
118	0.5	1.50E-07	2.10E-07	1.90E-07	1.34E-04
119	0.25	1.44E-06	1.84E-06	1.68E-06	7.50E-04
120	0.12	1.09E-05	1.36E-05	1.25E-05	3.46E-03
121	0.12	6.89E-05	8.42E-05	7.80E-05	1.32E-02
122	0.19	3.64E-04	4.37E-04	4.08E-04	4.25E-02
					Continued.

Table 7.4 (Continued)

K	DJX	BSP	CDP	SDP	SEP
123	0.2	1.62E-03	1.92E-03	1.80E-03	1.16E-01
124	0.2	6.16E-03	7.14E-03	6.76E-03	2.70E-01
125	0.36	2.01E-02	2.28E-02	2.18E-02	5.49E-01
126	0.33	5.64E-02	6.32E-02	6.06E-02	9.87E-01
127	0.47	0.13820666	0.15264987	0.14706735	1.59E+00
128	0.67	0.29835326	0.32491481	0.31471788	2.35E+00
129	0.99	0.57327787	0.61628186	0.59987668	3.21E+00
130	1.35	0.991955209	1.054018079	1.030477153	4.15E+00
131	2.34	1.56502375	1.64602608	1.61545363	5.12E+00
132	3.05	2.28076136	2.37795437	2.34142125	6.11E+00
133	4.95	3.11030108	3.21944034	3.17854591	7.10E+00
134	6.37	4.01844649	4.1352184	4.09156075	8.10E+00
135	17.9	4.97375236	5.09476035	5.04958194	9.10E+00

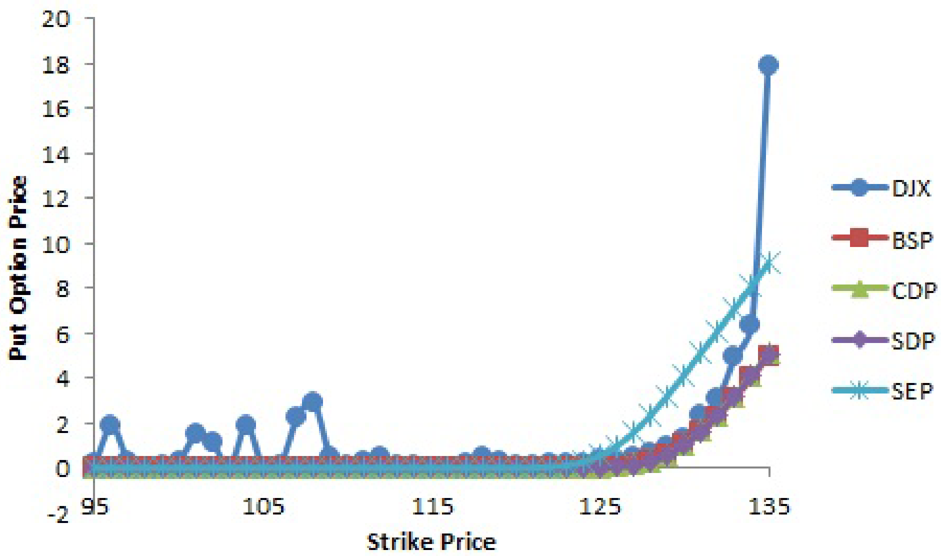

Figure 7.2: Comparison of real-market put option prices of DJX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

The simulation results for put options are also shown graphically in figure 7.2. The results of the sum of squared errors of the put option prices obtained from the four models are shown in table 7.5.

Table 7.5: Sums of squared errors of put option prices obtained by four different models for the 1/100 Dow Jones Industrial Average option prices

Model	BSP	CDP	SDP	SEP
Sum of Squared Errors	203.0267516	198.6196425	200.2528933	144.4874479

Obviously, for put option prices, our model also gives better agreement with the real-market data.

Furthermore, our extended option pricing model can perform even better results. For example, if we set $\sigma_S = 0.25$ and keep other parameters value unchanged, the sum of squared errors of call options reduce to 423.7833 which is a better outcome. However, for put option prices, the sum of squared errors will increase to 184.3575. This situation can happens since the σ_S that we set at the first time is not the accurate stock volatility and not the real implied volatility. Mathematically, we set the value of σ_S to 0.1 just for the mathematical testing reason. Consequently, in the real world, by using more realistic parameter values, our model can perform a better results. From the data, the figures and the numerical results presented, our extended option pricing model has better performance than previous models.

7.2.2 S&P 500 (ˆGSPC)

The S&P 500 or Standard and Poor 500 is an index based on 500 large-cap common stocks in the United States and has been published since 1957. It is purposed to be a significant indicator of equities in the United States and may reflect the characteristics of rate of returns for the large cap market; as a result, the S&P 500 is one of the most generally accepted equity indices after the Dow Jones Industrial Average.

We will perform the simulation of our option pricing by setting the parameters at the values as shown in table 7.6.

Table 7.6: Parameters values used in the simulation of the Standard and Poor 500 option prices

$S = 1369.63$	$\theta_\delta = 0.59$	$\theta_\xi = 0.79$	$\kappa_1(0) = 0.20$
$r = 0.053$	$\delta(0) = 0.02$	$\xi(0) = 0.062696$	$\kappa_2(0) = 0.20$
$T = 78/365$	$\mu_\delta = 0.0237$	$\mu_\xi = 0.067159$	$\kappa_3(0) = 0.20$
$\sigma_S = 0.12$	$\sigma_{\delta 1} = 0.01$	$\sigma_{\xi 1} = 0.05$	$\mu_{\kappa_1} = 0.20$
	$\sigma_{\delta 2} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_2} = 0.20$
	$\sigma_{\delta 3} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_3} = 0.20$
			$\theta_{\kappa 1} = 0.45$
			$\theta_{\kappa 2} = 0.45$
			$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.40$
			$\sigma_{\kappa 2} = 0.40$
			$\sigma_{\kappa 3} = 0.40$

Some of the data in the table 7.6, such as the current index price (S) and the risk free rate of return (r), are acquired by the real data in the market as of March 2^{nd}, 2012. We simulate the data by testing at the different maturity of 78 days, the real-data of option prices expire on May 19^{th}, 2012, as to perform the middle range of the maturity. By the same approach, the σ_S is the first parameter to determine by finding the least sum of squared errors of the Black-Scholes model.

Table 7.7 shows the results obtained by four different models where BSC, CDC, SDC and SEC are the simulation results of the Black-Scholes model, the Constant dividend Black-Scholes-Merton model, the Stochastic dividend option pricing model and the Stochastic earning yield option pricing model, while SNP is the real market data of the Standard and Poor 500 call option prices. The figure of the simulation results confirms the argument of Rubinstein that the Black-Scholes model with or without constant dividend yield performs a very good approximation to the real market data of the S&P 500 index [82].

Table 7.7: Comparison of real-market call option prices of SNP with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	SNP	BSC	CDC	SDC	SEC
1050	312	331.4552396	325.6139715	329.4254726	323.180743
1100	275.51	282.0185813	276.1774108	279.9888594	273.7455448
1190	176.25	193.0976898	187.2750823	191.07591	1.85E+02
1275	104.5	111.150351	105.7118323	109.2813388	104.5182688
1300	87.5	88.889206	83.7983023	87.1531897	83.2759706
1305	86	84.6356013	79.6328258	82.9328434	79.2557163
1325	70.95	68.4611122	63.873635	66.9131278	6.41E+01
1330	63.9	64.653466	60.1848026	63.1492369	6.05E+01
1340	57.35	57.3551163	53.1401933	55.9436384	5.38E+01
1345	56	53.8729357	49.7921621	52.5101335	5.05E+01
					Continued.

Table 7.7 (Continued)

K	DJX	BSC	CDC	SDC	SEC
1350	53.5	50.5071491	46.5647679	49.1943131	4.74E+01
1355	51.6	47.2610269	43.4607641	45.9992475	4.45E+01
1360	44.45	44.1373758	40.4823977	42.9275302	4.16E+01
1365	41	41.1385031	37.63138322	39.9812467	3.88E+01
1370	39.5	38.26619566	34.90888836	37.16195599	3.62E+01
1375	36.6	35.52170365	32.31552526	34.47067767	3.37E+01
1380	35.15	32.90572725	29.85134671	31.90788053	3.13E+01
1385	29.95	30.41841528	27.51585249	29.47348534	2.90E+01
1390	27.6	28.05937082	25.30800309	27.16687275	2.68E+01
1400	22.85	23.72182583	21.2684922	22.9318923	2.29E+01
1420	14.7	16.5115176	14.621026	15.912333	1.62E+01
1425	13.3	14.9970462	13.2375101	14.4416612	1.48E+01
1430	14.6	13.5904939	11.9572275	13.0771205	1.35E+01
1435	10.45	12.2876871	10.7757053	11.8144546	1.23E+01
1440	9.3	11.0842327	9.6883045	10.6492077	1.11E+01
1450	7.25	8.9569907	7.7767601	8.592419	9.14E+00
1460	7	7.1709609	6.1838758	6.8688056	7.44E+00
1470	3.76	5.69E+00	4.8710142	5.4399801	6.00E+00
1475	3.35	5.05E+00	4.3078187	4.8242892	5.38E+00
1480	2.65	4.47E+00	3.8007334	4.2682885	4.81E+00
1490	2.1	3.48E+00	2.9376782	3.3177794	3.82E+00
1500	1.35	2.68E+00	2.2492391	2.5549456	3.01E+00
1520	0.85	1.55E+00	1.2818335	1.47337	1.83E+00
1525	0.67	1.34E+00	1.1072309	1.2765183	1.61E+00
1540	0.5	8.64E-01	0.7037245	0.8187924	1.08E+00
1550	0.35	6.36E-01	0.5142424	0.6020579	8.17E-01
1575	0.48	2.84E-01	0.2255683	0.2683115	3.96E-01
1600	0.2	1.20E-01	9.36E-02	0.1131274	1.84E-01

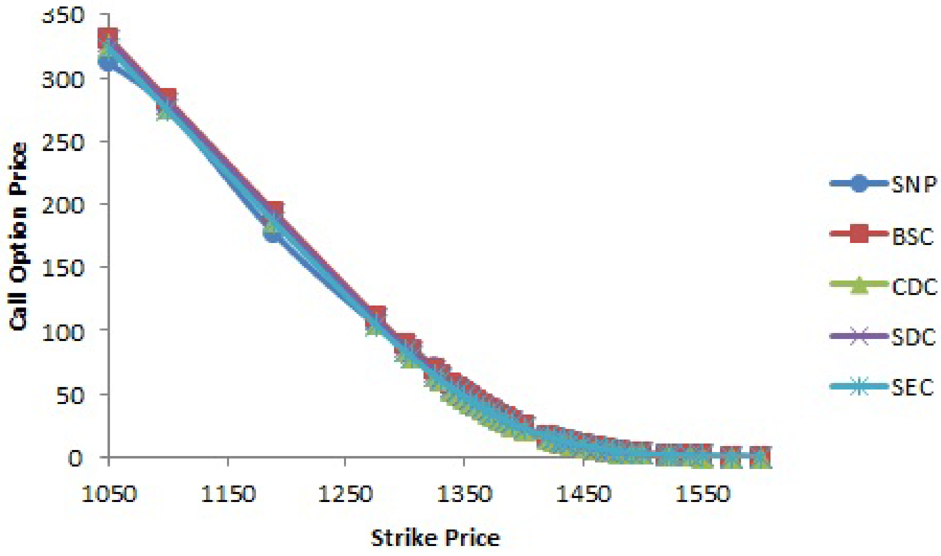

Figure 7.3: Comparison of real-market call option prices of SNP with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

By comparing the real market data of S&P 500 index options (SNP) and the results from our option pricing model (SEC), we notice that our result is very close to the real data graph as shown in figure 7.3. We calculate the sum of squared errors for all the four models in order to compare the effectiveness of the models in simulating the option prices.

Table 7.8: Sums of squared errors of call option prices obtained by four different models for the Standard and Poor 500 option prices

	BSC	CDC	SDC	SEC
Sum of Squared Errors	832.3042676	719.5150377	703.2344074	543.2178641

As shown in Table 7.8, our option pricing model taking into account stochastic earning yield has performed better for the approximation of option prices, even though, for the S&P 500 index options, it was argued that the Black-Scholes-Merton model was the best approximation.

Although study on the call prices has shown the superior performance of our option pricing model, the put option prices should be examined as well. By using the same parameter settings and the same approach, put option prices are obtained by using different models which are shown in table 7.9.

Table 7.9: Comparison of real-market put option prices of SNP with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	SNP	BSP	CDP	SDP	SEP
500	0.05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
550	0.15	0.00E+00	0.00E+00	0.00E+00	0.00E+00
600	0.35	0.00E+00	0.00E+00	0.00E+00	0.00E+00
625	0.1	0.00E+00	0.00E+00	0.00E+00	0.00E+00
650	0.35	0.00E+00	0.00E+00	0.00E+00	0.00E+00
675	0.4	0.00E+00	0.00E+00	0.00E+00	0.00E+00
700	0.3	0.00E+00	0.00E+00	0.00E+00	0.00E+00
725	0.4	0.00E+00	0.00E+00	0.00E+00	0.00E+00
730	0.45	0.00E+00	0.00E+00	0.00E+00	0.00E+00
740	0.5	0.00E+00	0.00E+00	0.00E+00	0.00E+00
750	0.4	0.00E+00	0.00E+00	0.00E+00	0.00E+00
775	0.6	0.00E+00	0.00E+00	0.00E+00	0.00E+00
					Continued.

Table 7.9 (Continued)

K	SNP	BSP	CDP	SDP	SEP
800	0.5	0.00E+00	0	0.00E+00	0.00E+00
825	1.65	0.00E+00	0	0.00E+00	0.00E+00
850	1.5	0.00E+00	0	0.00E+00	0.00E+00
860	1	0.00E+00	0	0.00E+00	0.00E+00
875	2.55	0.00E+00	0	0.00E+00	0.00E+00
900	1.4	0.00E+00	0	0.00E+00	0.00E+00
925	2.15	0.00E+00	0	0.00E+00	0.00E+00
930	1.95	0.00E+00	0	0.00E+00	-1.00E-07
950	1.85	0.00E+00	0	0.00E+00	0.00E+00
975	2	0.00E+00	0	0.00E+00	1.00E-07
990	2.9	0.00E+00	0.00E+00	1.00E-07	3.00E-07
1,000.00	2.9	0.00E+00	0.00E+00	0.00E+00	8.00E-07
1,025.00	2.5	4.00E-07	5.00E-07	5.00E-07	6.80E-06
1,040.00	3	1.40E-06	2.20E-06	1.80E-06	2.35E-05
1,050.00	3.5	3.60E-06	5.50E-06	4.60E-06	5.20E-05
1,070.00	3.66	2.14E-05	3.15E-05	2.63E-05	2.31E-04
1,075.00	3.85	3.28E-05	4.78E-05	3.99E-05	3.29E-04
1,080.00	8	4.97E-05	7.20E-05	6.02E-05	4.66E-04
1,090.00	4.5	1.11E-04	1.59E-04	1.33E-04	9.15E-04
1,100.00	4.6	2.39E-04	3.39E-04	2.85E-04	1.75E-03
1,110.00	6.05	5.01E-04	7.01E-04	5.93E-04	3.25E-03
1,120.00	6	1.02E-03	1.41E-03	1.19E-03	5.90E-03
1,125.00	5	1.43E-03	1.97E-03	1.68E-03	7.87E-03
1,130.00	6.7	2.00E-03	2.74E-03	2.34E-03	1.04E-02
1,140.00	6.75	3.84E-03	5.19E-03	4.44E-03	1.80E-02
1,150.00	6.6	7.14E-03	9.56E-03	8.21E-03	3.04E-02
1,160.00	10.1	1.29E-02	1.71E-02	1.48E-02	5.01E-02
1,170.00	8.52	2.28E-02	2.99E-02	2.59E-02	8.08E-02
					Continued.

Table 7.9 (Continued)

K	SNP	BSP	CDP	SDP	SEP
1,175.00	8	3.00E-02	3.91E-02	3.40E-02	1.02E-01
1,180.00	8.5	3.92E-02	5.09E-02	4.42E-02	1.28E-01
1,190.00	10.8	6.58E-02	8.44E-02	7.37E-02	1.97E-01
1,200.00	9.75	0.1076277	0.1367979	0.1200084	0.298982
1,210.00	10.7	0.1720886	0.2165906	0.1908185	0.4444844
1,220.00	11.45	0.2689916	0.3353048	0.2966692	0.6485009
1,225.00	11.91	0.3335345	0.4137893	0.3668923	0.7779696
1,230.00	12.65	0.4113395	0.5079228	0.4513185	0.929128
1,240.00	13.05	0.6158127	0.7534039	0.6722999	1.3080232
1,250.00	14.69	0.9032161	1.0950649	0.9813395	1.8104695
1,260.00	16	1.2987768	1.5607777	1.4046092	2.4652463
1,270.00	17.1	1.8322242	2.1829069	1.972752	3.3042824
1,275.00	17.95	2.161136	2.5638873	2.3218918	3.8035988
1,280.00	19	2.5375983	2.9979458	2.7206219	4.3620767
1,290.00	20.8	3.4527382	4.0458117	3.6867018	5.6748919
1,300.00	22	4.618438	5.3688043	4.9121897	7.2797476
1,310.00	23.8	6.0772786	7.0102583	6.4397731	9.2132578
1,315.00	29.3	6.9300736	7.963848	7.3302376	10.3142818
1,320.00	27.2	7.872193	9.012963	8.3121489	11.5103782
1,325.00	29.3	8.9087912	10.162584	9.3905748	12.8054485
1,330.00	29.7	10.044834	11.4174406	10.5703729	14.2031346
1,340.00	32.6	12.6338633	14.2602103	13.2521534	17.3194168
1,345.00	37.5	14.0953717	15.8558681	14.7623375	19.0436982
1,350.00	36.1	15.6732741	17.5721629	16.3902061	20.8819132
1,355.00	37.2	17.3708419	19.4118491	18.1388305	22.8359455
1,360.00	41	19.1908798	21.3771717	20.0108022	24.9072597
1,365.00	43.2	21.1356961	23.46984622	22.0082077	27.09689322
1,370.00	43.4	23.20707866	25.69104136	24.13260699	29.40544718
					Continued.

Table 7.9 (Continued)

K	SNP	BSP	CDP	SDP	SEP
1,375.00	43.1	25.40627565	28.04136726	26.38501767	31.83308374
1,450.00	114.3	72.9969027	77.6579421	74.662099	81.4550259
1,550.00	251	163.549805	169.2692114	165.5455249	172.0048646
1,600.00	239.5	212.4707273	218.2854323	214.4934884	220.8086605

Figure 7.4 shows the graph of the put option prices as functions of the strike price for the four models, together with the S&P 500 index option prices. From the figure, we can see that the difference among different models and the real-market data are not outstandingly obvious, since the S&P 500 index option prices are well known to be approximated by the Black-Scholes model. We make a further investigation by computing the sum of squared errors shown in the table 7.10. The numerical results also obviously indicate that our extended option pricing model is the best model compared among the same type of mathematical models.

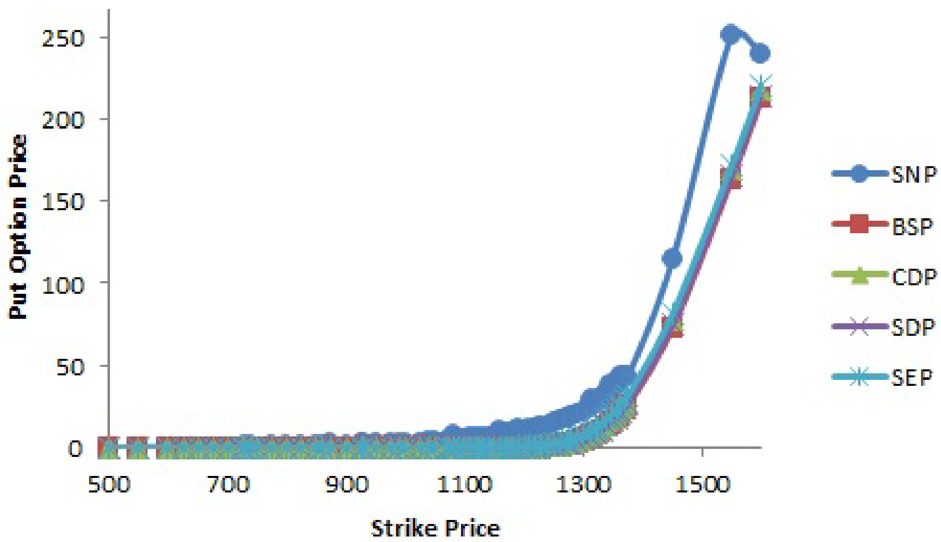

Figure 7.4: Comparison of real-market put option prices of SNP with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

Table 7.10: Sums of squared errors of put option prices obtained by four different models for the Standard and Poor 500 option prices

Model	BSP	CDP	SDP	SEP
Sum of Squared Errors	18910.14272	16307.61648	17937.6114	13820.58512

Consequently, in the next section we will investigate more by applying the other European option data.

7.2.3 Russell 2000 (ˆRUT)

The Russell 2000 is used as a benchmark for small cap stocks, including approximately 2000 smallest companies, in the United States. The Russell 2000 index option is an European option. We will perform the same investigation of the option prices through simulation using different option pricing models. We set the initial time on 2^{nd} March of 2012 and select a long range of maturity with the expiration date on December 21^{st}, 2012. Table 7.11 gives the parameters setting used for the simulation.

Again, some of the data above are from the real-world data which can be acquired from the professional source such as the wall street and yahoo finance website. With the same steps of investigation, we derive the model results as given in table 7.12.

Table 7.11: Parameters values used in the simulation of the Russell 2000 option prices

$S = 802.42$	$\theta_\delta = 0.70$	$\theta_\xi = 0.45$	$\kappa_1(0) = 0.20$
$r = 0.053$	$\delta(0) = 0.0142$	$\xi(0) = 0.023496$	$\kappa_2(0) = 0.20$
$T = 285/365$	$\mu_\delta = 0.0129$	$\mu_\xi = 0.015029$	$\kappa_3(0) = 0.20$
$\sigma_S = 0.21$	$\sigma_{\delta 1} = 0.01$	$\sigma_{\xi 1} = 0.05$	$\mu_{\kappa_1} = 0.20$
	$\sigma_{\delta 2} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_2} = 0.20$
	$\sigma_{\delta 3} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_3} = 0.20$
			$\theta_{\kappa 1} = 0.45$
			$\theta_{\kappa 2} = 0.45$
			$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.40$
			$\sigma_{\kappa 2} = 0.40$
			$\sigma_{\kappa 3} = 0.40$

Table 7.12: Comparison of real-market call option prices of RUT with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	RUT	BSC	CDC	SDC	SEC
200	471.6	610.5277906	601.679963	607.795101	537.3088868
210	501.1	600.9331802	592.0853526	598.2004906	527.7142765
220	438.6	591.3385696	582.490742	588.60588	518.1196661
230	487.4	581.7439592	572.8961316	579.0112696	508.5250563
240	493.3	572.1493488	563.3015212	569.4166592	498.9304477
250	436.5	562.5547384	553.7069108	559.8220488	489.3358419
260	441.9	552.9601278	544.1123002	550.2274382	479.7412424
270	432.6	543.3655173	534.5176896	540.6328276	470.1466571
280	455.3	533.7709069	524.9230792	531.0382172	460.552101
					Continued.

Table 7.12 (Continued)

K	RUT	BSC	CDC	SDC	SEC
290	443.4	524.1762965	515.328469	521.4436069	450.9576017
300	424.3	514.5816863	505.7338588	511.8489967	441.3632077
330	398.9	485.797859	476.9500332	483.0651701	412.5817565
350	370	466.6086589	457.76084	463.8759724	393.3980205
400	344	418.6364011	409.788813	415.903796	345.4940093
450	330.7	370.6744207	361.8295486	367.9427563	297.8735597
500	317.9	322.7853149	313.9577817	320.0595578	251.111987
525	295.4	298.9319186	290.1288519	296.2144106	228.3579959
550	234.9	275.2136088	266.4555011	272.5111435	206.2341546
575	187.7	251.7227838	243.0403171	249.0454395	184.9159119
600	236.3	228.5854436	220.0206804	225.9469091	164.5771767
625	210.49	205.9595256	197.565637	203.3769396	145.3777397
650	189.14	184.0279177	175.8671962	181.5211578	1.27E+02
675	169.75	162.9867663	155.1270496	160.5772458	1.11E+02
700	151.27	143.0307977	135.5407786	140.7399374	9.58E+01
725	132.52	124.3379583	117.2820803	122.1855644	8.21E+01
740	130.38	113.7915048	107.0234656	111.7304645	7.47E+01
750	123.4	107.0557083	100.4893962	105.0584968	7.00E+01
760	98.14	100.563213	94.2054717	98.6318359	6.55E+01
770	91.87	94.3188072	88.17553068	92.45496252	6.12E+01
775	86.53	91.29079834	85.25665442	89.46125703	5.92E+01
780	84.7	88.32604989	82.40213545	86.53111553	5.72E+01
790	78.95	82.58726643	76.88658393	80.86229262	5.34E+01
800	73.35	77.10356164	71.62893854	75.44926865	4.97E+01
810	67.91	71.87484989	66.62807131	70.29163064	4.63E+01
820	62.53	66.89990054	61.88172355	65.38782692	4.31E+01
825	59.68	64.50690315	59.60299669	63.03033189	4.15E+01
830	58.06	62.17639583	57.38657679	60.73522976	4.00E+01
					Continued.

Table 7.12 (Continued)

K	RUT	BSC	CDC	SDC	SEC
840	53.75	57.70099989	53.1383335	56.33020724	3.72E+01
850	49	53.46943706	49.13180552	52.16820493	3.45E+01
875	44.8	4.39E+01	4.01E+01	4.28E+01	2.85E+01
900	36	3.58E+01	3.25E+01	3.48E+01	2.34E+01
950	20.5	2.31E+01	20.7727754	22.4338151	1.56E+01
1,000.00	11.1	1.45E+01	12.8751593	14.0265458	1.02E+01

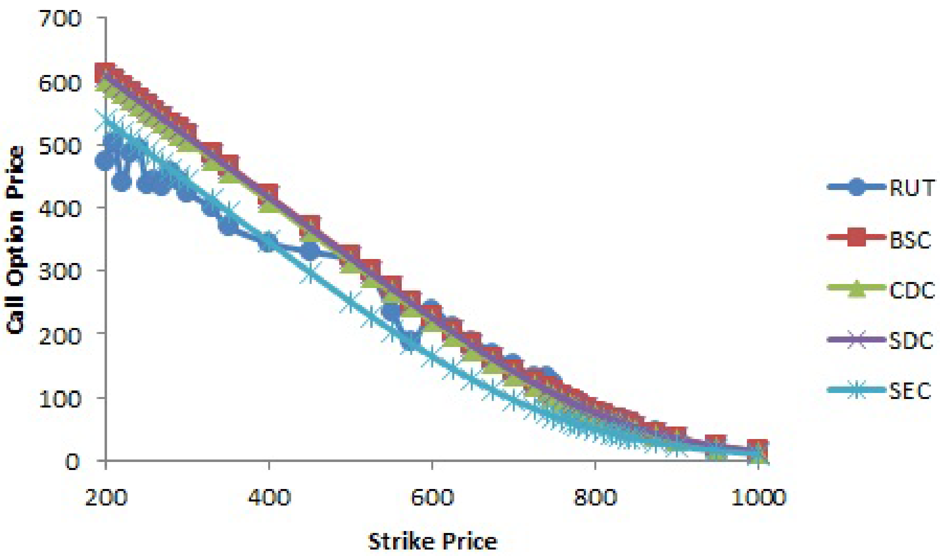

Figure 7.5: Comparison of real-market call option prices of RUT with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

The results in table 7.12 are also shown graphically by figure 7.5. The results have shown that our model yields results closest to the real-market data; while results of the other three models are in the same range of prices but situate out of range to the real market data mostly overpriced. Numerically, in table 7.13, we use the sum of squared errors to verify the effectiveness of our option pricing model in comparison to the other three models. Obviously, as shown in table 7.13, our model has the best performance in terms of the least squared errors. We also simulate the put option prices by the four models for the Russell 2000 by applying the same approach in order to compare the effectiveness of the models, and the results are shown in table 7.14.

Table 7.13: Sums of squared errors of call option prices obtained by four different models for the Russell 2000 option prices

	BSC	CDC	SDC	SEC
Sum of Squared Errors	159888.5865	134986.2355	151697.9344	64821.92933

Table 7.14: Comparison of real-market put option prices of RUT with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	RUT	BSP	CDP	SDP	SEP
200	1.25	0.00E+00	0.00E+00	0.00E+00	0.00E+00
210	1.05	0.00E+00	0.00E+00	0.00E+00	0.00E+00
220	1	0.00E+00	0.00E+00	0.00E+00	0.00E+00
230	1.15	0.00E+00	0.00E+00	0.00E+00	0.00E+00
240	1.1	0.00E+00	0.00E+00	0.00E+00	2.00E-07
250	1.2	0.00E+00	0.00E+00	0.00E+00	6.00E-07
260	1	0.00E+00	0.00E+00	0.00E+00	1.80E-06
270	1	1.00E-07	0.00E+00	0.00E+00	4.70E-06
280	1.3	1.00E-07	0.00E+00	0.00E+00	1.17E-05
290	1.2	1.00E-07	2.00E-07	1.00E-07	2.79E-05
300	1.8	3.00E-07	4.00E-07	3.00E-07	6.29E-05
310	5	7.00E-07	9.00E-07	8.00E-07	1.35E-04
320	2.05	1.80E-06	2.50E-06	2.10E-06	2.77E-04
330	4.8	4.60E-06	6.40E-06	5.30E-06	5.45E-04
340	4	1.11E-05	1.51E-05	1.26E-05	1.03E-03
350	3.18	2.53E-05	3.40E-05	2.84E-05	1.88E-03
360	5	5.48E-05	7.31E-05	6.14E-05	3.33E-03
380	4.06	2.28E-04	2.99E-04	2.53E-04	9.49E-03
390	4.5	4.40E-04	5.73E-04	4.87E-04	1.54E-02
400	4.3	8.20E-04	1.06E-03	9.04E-04	2.43E-02
430	21	4.42E-03	5.60E-03	4.83E-03	8.37E-02
440	13	7.34E-03	9.23E-03	8.00E-03	1.21E-01
450	9.51	1.19E-02	1.48E-02	1.29E-02	1.73E-01
460	9.7	1.88E-02	2.33E-02	2.04E-02	2.42E-01
470	10.5	2.91E-02	3.59E-02	3.15E-02	3.34E-01
480	36.5	4.42E-02	5.41E-02	4.76E-02	4.53E-01
490	10.8	6.57E-02	8.00E-02	7.06E-02	6.06E-01
500	11.7	9.58E-02	1.16E-01	1.03E-01	8.00E-01
					Continued.

Table 7.14 (Continued)

K	RUT	BSP	CDP	SDP	SEP
525	16.3	2.29E-01	2.74E-01	2.44E-01	1.52E+00
550	16.79	4.97E-01	5.87E-01	5.27E-01	2.68E+00
575	23.9	9.93E-01	1.16E+00	1.05E+00	4.46E+00
600	23	1.84E+00	2.13E+00	1.94E+00	7.04E+00
625	26.8	3.20268	3.656619	3.3527836	1.06E+01
650	33.5	5.2575981	5.9447042	5.4835278	1.53E+01
660	36.9	6.3177893	7.1169957	6.5800083	1.75E+01
675	40.6	8.2029731	9.191084	8.5261422	2.13E+01
690	44.74	10.4791487	11.6809932	10.8710111	2.55E+01
700	47	12.2335305	13.591339	12.6753598	2.87E+01
710	50.4	14.1903596	15.7144975	14.6853417	3.20E+01
720	48.23	16.3598175	18.0601539	16.9109648	3.56E+01
725	54.58	17.5272173	19.3191669	18.107513	3.75E+01
730	52.8	18.7511728	20.6369651	19.3612893	3.95E+01
740	58.4	21.3726796	23.452468	22.0443289	43.56440227
750	63.1	24.2314935	26.513009	24.9669716	47.89158687
760	64.39	27.3336086	29.8236949	28.1349211	52.4582257
770	67.96	30.6838132	33.3883643	31.5526581	57.26206491
775	71.88	32.4531095	35.2667932	33.3562578	59.75199723
780	75.76	34.2856665	37.20957965	35.2234217	62.30000498
790	80.09	38.14149363	41.28863873	39.14920942	67.56816778
800	84.56	42.25239924	45.62560374	43.33079584	73.06196774
810	89.09	46.61829789	50.21934691	47.76776824	78.776186
820	93.81	51.23795894	55.06760955	52.45857492	84.70504548
825	94.49	53.64226675	57.58618789	54.89838509	87.74802621
830	98.88	56.10906463	60.16707319	57.40058816	90.84228632
840	102.57	61.22827929	65.5134405	62.59017624	97.18124077
850	107.78	66.59132706	71.10152312	68.02278453	103.7149052
					Continued.

Table 7.14 (Continued)

K	RUT	BSP	CDP	SDP	SEP
875	109.1	81.03018233	86.08824932	82.63033661	120.8527433
900	123.2	96.85962365	102.4356645	98.6183978	139.0437667
950	166.1	132.1978021	138.6885976	134.2344993	178.1081548
1,000.00	206.6	171.5444469	178.7640337	173.8002822	219.9983243

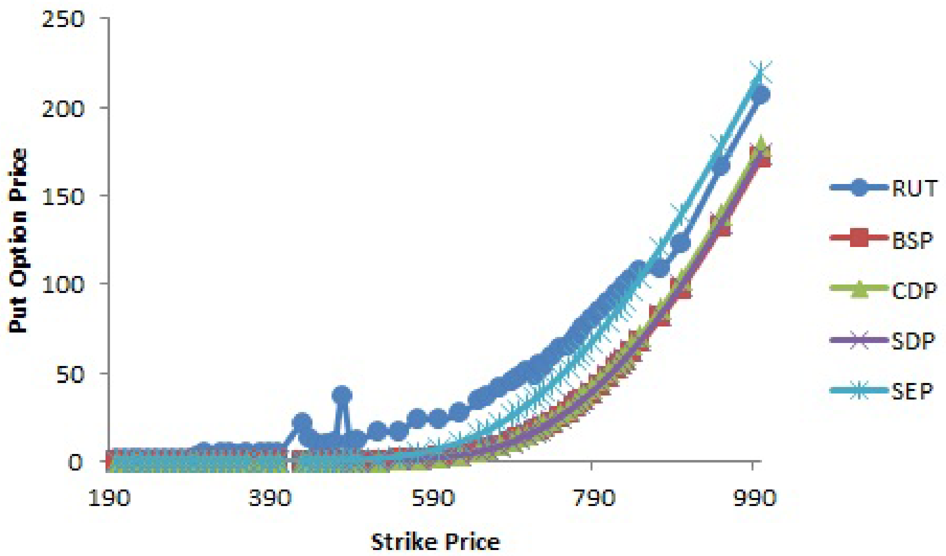

Figure 7.6: Comparison of real-market put option prices of RUT with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

As we can see from the figure 7.6, the three existing models, the Black-Scholes model, the Black-Scholes-Merton model with constant dividend and the stochastic dividend yield option pricing model, always yield underpriced option values; while our model yields underpriced option value for some strike values but overpriced option value for some other strike values. By this property, our model performs a better characteristic of fair value for option pricing model.

In table 7.15, we give the sum of squared errors for all the four models. The results show the superior performance of our option pricing model taking into account the stochastic earning yield.

Table 7.15: Sums of squared errors of put option prices obtained by four different models for the Russell 2000 option prices

Model	BSP	CDP	SDP	SEP
Sum of Squared Errors	41582.13346	35771.67166	39656.74274	9026.468501

7.2.4 NASDAQ-100 (ˆNDX)

In this section we will examine our model using the European option of NASDAQ-100. The NASDAQ-100 is an index of stock market of the top 100 largest non-financial firms on the list of NASDAQ. This index is computed by a modified capitalization-weighted method. The weight of each company is based on the market capitalization of the company. This index does not include financial companies, and does contain some companies incorporated outside the United States. Because of these factors, NASDAQ-100 is different from the Dow Jones Industrial Average and the S&P 500 Index which we already explore in previous sections.

We perform the empirical test by utilizing the same approach as previous sections. In this test, we select a very long maturity time. The option price data is acquired on March 2^{nd}, 2012 and will be expired at the close of Friday, December 20^{th}, 2013. The parameter values used in the test are shown in table 7.16.

As before, the parameter σ_S is determined first to yield the least sum of squared errors for the Black-Scholes model. The same approach is then used to simulate the option prices by using different models. The results are shown in table 7.17.

Table 7.16: Parameters values used in the simulation of the NASDAQ-100 option prices

$S = 2641.58$	$\theta_\delta = 0.22$	$\theta_\xi = 0.40$	$\kappa_1(0) = 0.20$
$r = 0.053$	$\delta(0) = 0.0093$	$\xi(0) = 0.084459$	$\kappa_2(0) = 0.20$
$T = 658/365$	$\mu_\delta = 0.0079$	$\mu_\xi = 0.074794$	$\kappa_3(0) = 0.20$
$\sigma_S = 0.07$	$\sigma_{\delta 1} = 0.01$	$\sigma_{\xi 1} = 0.05$	$\mu_{\kappa_1} = 0.20$
	$\sigma_{\delta 2} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_2} = 0.20$
	$\sigma_{\delta 3} = 0.01$	$\sigma_{\xi 2} = 0.05$	$\mu_{\kappa_3} = 0.20$
			$\theta_{\kappa 1} = 0.45$
			$\theta_{\kappa 2} = 0.45$
			$\theta_{\kappa 3} = 0.45$
			$\sigma_{\kappa 1} = 0.40$
			$\sigma_{\kappa 2} = 0.40$
			$\sigma_{\kappa 3} = 0.40$

Table 7.17: Comparison of real-market call option prices of NDX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	NDX	BSC	CDC	SDC	SEC
1,500.00	1,072.20	1278.264088	1234.345916	1382.239952	853.1077106
1,600.00	799.7	1187.376361	1143.458189	1291.352226	771.2096668
1,700.00	725.3	1096.488632	1052.570461	1200.464509	692.9262461
1,800.00	654	1005.600913	961.6827501	1109.576922	618.8516311
1,850.00	478.3	960.1570712	916.2389418	1064.133393	583.5486864
1,900.00	450	914.7133029	870.7952987	1018.690476	549.4688991
1,950.00	422.5	869.269788	825.3522063	973.2490979	516.6513988
2,000.00	395.7	823.8270789	779.9107743	927.8112891	485.1262486
2,100.00	459	732.9522959	689.0483192	836.9668866	426.0284974
2,150.00	382.5	687.5323029	643.6486217	791.5826141	398.4721388
2,200.00	342.26	642.1439787	598.3034094	746.2526072	372.2415166
					Continued.

Table 7.17 (Continued)

K	NDX	BSC	CDC	SDC	SEC
2,250.00	242.08	596.820169	553.064525	701.015431	347.3254528
2,300.00	339.85	551.618577	508.018769	655.929248	323.7061771
2,350.00	300	506.633228	463.300937	611.077288	301.3600329
2,400.00	250.4	462.006647	419.105659	566.572794	280.258199
2,450.00	194.7	417.940309	375.6947636	522.562517	260.367415
2,500.00	282.15	374.70023	333.396752	479.227848	241.6506891
2,550.00	328.1	332.6147451	292.5959131	436.7827887	224.0679765
2,600.00	295.1	292.0625303	253.7105845	395.4684089	207.5768291
2,700.00	270	217.1851213	183.3405876	317.2742208	177.6909544
2,800.00	195.7	153.1091059	125.0668999	246.7088542	151.6269251
2,900.00	95.4	101.8458325	80.2089734	185.494452	1.29E+02
3,000.00	126.5	63.7382556	48.257788	134.6417138	1.09E+02

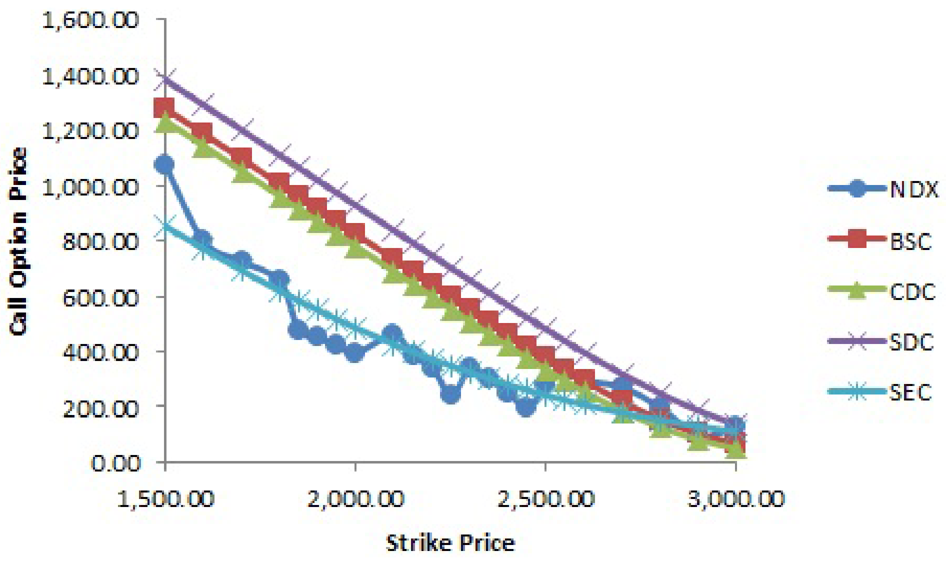

Figure 7.7: Comparison of real-market call option prices of NDX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

As shown by figure 7.7, the classic Black-Scholes model overprices the option values. The constant dividend yield Black-Scholes-Merton model and the stochastic dividend yield option pricing model also yield results far beyond the actual option prices. On the contrary, our model taking into account stochastic earing yield gives option prices very close to the actual prices with both underpriced and overpriced call option values as shown in figure 7.7.

Table 7.18: Sums of squared errors of call option prices obtained by four different models for the NASDAQ-100 option prices

	BSC	CDC	SDC	SEC
Sum of Squared Errors	1868401.821	1449013.061	3186173.982	139719.318

The results of the sum of squared errors for different models as shown in table 7.18 also confirm the superior performance of our extended model over the other models. The sum of squared error of our call option pricing model is much less than those by the other pricing models.

The put option prices of NASDAQ-100 are also compared to the simulated prices obtained by different option pricing models. The simulation results are shown in tables 7.19 and figure 7.8.

Table 7.19: Comparison of real-market put option prices of NDX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

K	NDX	BSP	CDP	SDP	SEP
1,000.00	15.8	0.00E+00	0.00E+00	0.00E+00	2.69E-01
1,250.00	30.9	0.00E+00	0.00E+00	0.00E+00	2.81E+00
1,500.00	63.7	0.00E+00	0.00E+00	0.00E+00	1.39E+01
1,600.00	79.8	0.00E+00	0.00E+00	1.00E-06	2.29E+01
1,700.00	119.1	0.00E+00	1.00E-06	1.20E-05	3.55E+01
1,800.00	110	7.00E-06	1.60E-05	1.52E-04	5.23E+01
1,850.00	113.9	2.90E-05	7.20E-05	4.87E-04	6.24E+01
1,900.00	195.9	1.25E-04	2.93E-04	1.43E-03	7.38E+01
1,950.00	202.2	4.73E-04	1.06E-03	3.92E-03	8.64E+01
2,000.00	155.8	1.63E-03	3.50E-03	9.97E-03	1.00E+02
2,050.00	168.4	5.09E-03	1.05E-02	2.38E-02	1.16E+02
2,100.00	175	1.46E-02	2.88E-02	5.33E-02	1.32E+02
2,150.00	190.4	3.84E-02	7.29E-02	1.13E-01	1.50E+02
2,200.00	228	9.40E-02	0.171585	2.27E-01	1.69E+02
2,250.00	219.2	2.14E-01	0.376565	4.33E-01	1.90E+02
2,300.00	333.4	4.56E-01	0.774673	7.91E-01	2.12E+02
2,350.00	240	9.15E-01	1.500705	1.38E+00	2.35E+02
2,400.00	280.2	1.73E+00	2.749291	2.32E+00	2.59E+02
2,450.00	296.19	3.11E+00	4.78226	3.76E+00	2.85E+02
2,500.00	315.93	5.31E+00	7.92811	5.87E+00	3.11E+02
2,550.00	319.1	8.67E+00	12.571135	8.86E+00	3.39E+02
2,600.00	527.5	1.36E+01	19.12967	1.30E+01	3.68E+02
2,700.00	690	2.96E+01	3.96E+01	2.57E+01	4.29E+02
2,800.00	445.6	5.64E+01	7.23E+01	4.60E+01	4.94E+02
2,900.00	806.1	9.60E+01	1.18E+02	7.57E+01	5.62E+02
3,000.00	864.34	1.49E+02	1.77E+02	1.16E+02	6.34E+02

From the table 7.19 and figure 7.8, it is clear that the Black-Scholes model, the constant dividend yield Black-Scholes-Merton model and the stochastic dividend yield model give option prices in the same range, but the results are all out of the range of the actual market option prices; this situation causes the underpriced option values. Whereas, as shown in figure 7.8, our option pricing model taking into account stochastic earning yield gives fair prices of options with the smallest sum of squared errors compared to the other models and provides both underpriced and overpriced options for the investors to consider at each point of time.

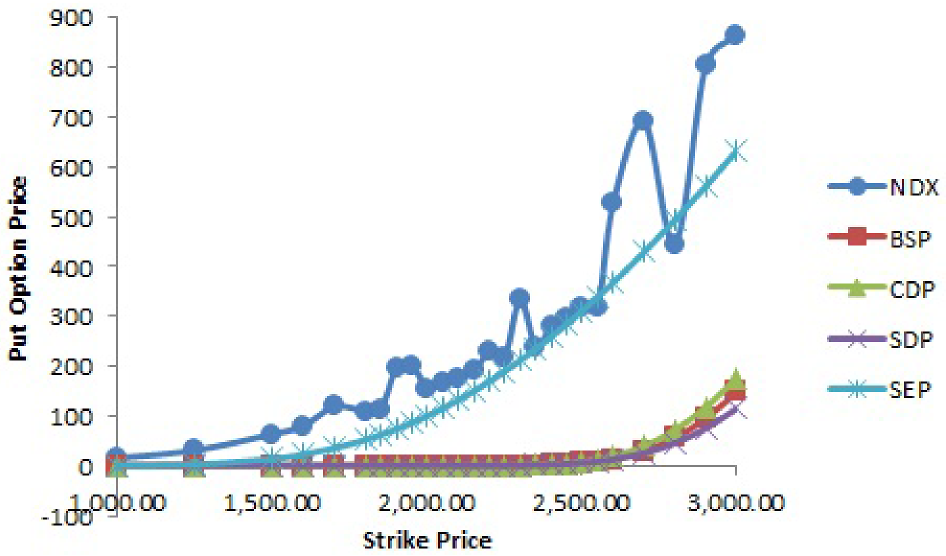

Figure 7.8: Comparison of real-market put option prices of NDX with simulation results obtained from four different models: BSC, CDC, SDC and SEC.

Table 7.20: Sums of squared errors of put option prices obtained by four different models for the NASDAQ-100 option prices

Model	BSP	CDP	SDP	SEP
Sum of Squared Errors	2741856.177	2633634.338	2831656.858	286117.471

7.3 An Application of Option Pricing Model

Currently, there are many option valuation models available. Some of the models are widely used. For example, in the Australian market, the Black-Scholes model and the binomial model are the main models for the investors to determine the fair price of the options.

In this section, we will show an application of our option pricing model by comparing the model results with the real-market data. We select the stock market index: Dow Jones Industrial Average, to examine the trading by using our option pricing model.

As of Friday, March 02, 2012, we presume that we decide to take a long position of an option. The first step is to examine the trend of Dow Jones Industrial Average index, acquired from http://www.google.com/finance. By checking the 14 days historical data of the Dow Jones Industrial Average index, we derive the data as shown in table 7.21.

Table 7.21: Historical data of Dow Jones Industrial Average index from Feb 16, 2012 to Mar 1, 2012

Date	Open	High	Low	Close	Volume
Mar 1, 2012	12,952.29	13,032.67	12,943.06	12,980.30	139,674,364
Feb 29, 2012	13,005.42	13,055.75	12,929.66	12,952.07	182,456,880
Feb 28, 2012	12,976.74	13,021.51	12,952.82	13,005.12	114,493,947
Feb 27, 2012	12,981.13	13,027.52	12,882.59	12,981.51	143,531,006
Feb 24, 2012	12,981.20	13,013.82	12,950.59	12,982.95	89,441,680
Feb 23, 2012	12,937.08	12,996.08	12,882.67	12,984.69	120,481,608
Feb 22, 2012	12,966.22	12,977.91	12,914.83	12,938.67	124,201,906
Feb 21, 2012	12,949.34	13,005.04	12,926.11	12,965.69	164,782,123
Feb 17, 2012	12,903.33	12,967.92	12,903.25	12,949.87	234,650,437
Feb 16, 2012	12,779.81	12,914.00	12,779.58	12,904.08	134,453,922
Feb 15, 2012	12,864.65	12,899.47	12,753.62	12,780.95	127,559,614

Also we can get the chart of historical data from http://www.google.com/finance as shown in figure 7.9. The trader might consider that the trend of the data is going up; and as a result, call options should be a choice for the investors.

Figure 7.9: Historical Chart of ˆDJX from Feb 16, 2012 to Mar 1, 2012

By implementing our call option pricing model taking into account stochastic earning yield, the model results are shown together with real-market data of option prices in figure 7.10.

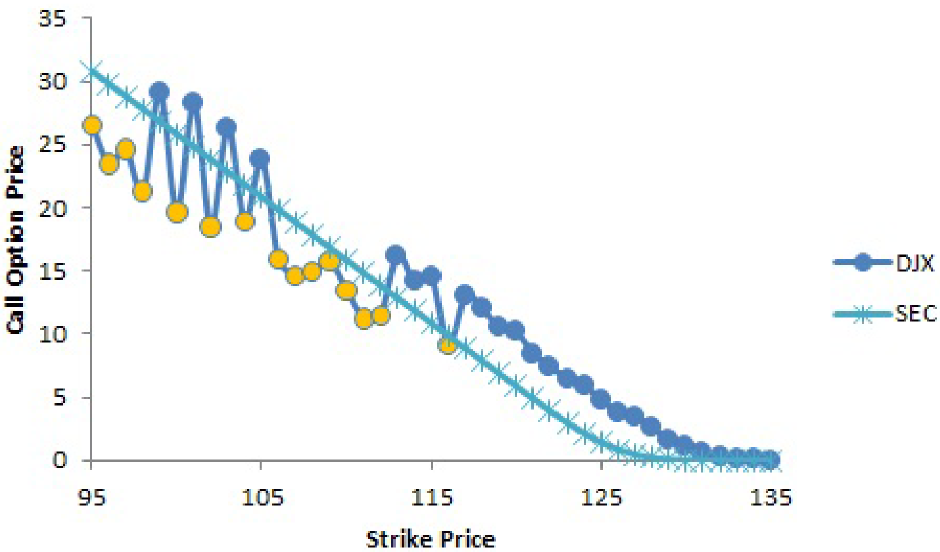

Figure 7.10: Comparison of option values between the real data ˆDJX and the data obtained by the option pricing model with stochastic earning yield

The graphs in figure 7.10 indicate that the market call option values are under-priced. Thus, the graph suggests the investors to purchase the call options at the strike prices shown in the gold dots. With trend consideration of the Dow Jones Industrial Average index, for example, consider the case that the traders decide to take a long position of the call option with the strike price of 100, which will expire at the close of Friday, March 16, 2012, assuming to be able to purchase at the date of Mar 2, 2012. From table 7.21, the traders must purchase the options with the contract multiplier, in this case the contract multiplier is \$100; therefore, the total payment for these options is \$1967 (assuming no transaction costs and fees).

Table 7.22: Dow Jones Industrial Average index price at Friday Mar 16, 2012

Date	Open	High	Low	Close	Volume
Mar 16, 2012	13,253.51	13,289.08	13,231.45	13,232.62	392,622,440

On Friday, March 16, 2012, from the table 7.22, the price of 1/100 Dow Jones Industrial Average (ˆDJX) is \$132.33. By exercising the call option, the traders will receive a cash settlement amount of $\big[(132.33 - 100) \times \$100)\big] - (\$1967)$, which is the profit of \$1266, assuming no transaction costs and taxes for the easy understanding.

In summary, the application shown above is one of the examples to show the application of the option pricing models to find the fair price and to make a judgement about the market value of the option. Nevertheless, there are still other pricing factors in the financial markets which affect the stock prices and option prices. Moreover, different expectations of parameters in the option pricing models will alter the result of fair value of the options. This implies that there may be various aspects held simultaneously on the option fair price at any one time.

7.4 Concluding Remarks

In this chapter, the application of the proposed model, the option pricing model taking into account stochastic dividend yield, stochastic market price of risk and stochastic earning yield, is investigated by comparing the model results with market option prices. Four stock index options are used for the investigation, including the 1/100 Dow Jones Industrial Average (ˆDJX), S&P 500 (ˆGSPC), Russell 2000 (ˆRUT) and NASDAQ-100 (ˆNDX). We compute the option prices using four different option pricing models, including the Black-Scholes model, the constant dividend yield Black-Scholes-Merton model, the stochastic dividend yield model and our stochastic earning yield model. Different times to maturity are used including short maturity, middle maturity, long maturity and very-long maturity. In order to be fair for every model, we first find the volatility of the stock σ_S which affects the sum of squared errors. Some of the parameters are acquired by the actual data in the professional sources such as Wall Street, Yahoo Finance and Google Finance; while the rest are determined by the appropriate assumptions. With the empirical approach, the option prices obtained by four different option pricing models are compared to the real-market data in both cases of call and put options. The results have shown that, by providing the right situation of the market stock with the right θ_δ and θ_ξ, our option pricing model yields the option prices closest to the actual data with the least sum of squared errors.

Consequently, our proposed option pricing model, the option pricing model taking into account stochastic dividend yield, stochastic market price of risk and stochastic earning yield, is a very outstanding candidate to enhance the performance of fair price judgement for the investors.

Chapter 8

Conclusions and Future Work

For more than three decades, an option is a derivative financial instrument which plays an important role in financial world. Since the classic Black-Scholes model was introduced in 1973, it has been well known that fair option prices can be calculated by a proper option pricing model and there are six major factors that influence the option price. The six factors are stock price (S), strike price (K), time to maturity (T), interest rate (r), volatility (σ) and dividend yield (δ).

In this thesis, we propose to include the P/E ratio as another factor to compute the option price. The idea of this is lightened from daily financial newspapers which always report the stock information of both dividend yield and P/E ratio. In reality, both dividend yield and P/E ratio have the immense influence to the underlying stock price and the future stock value or, in the other words, fair stock price can be determined by these parameters. As a matter of fact, the option price depends on the value of the stock price; consequently, both dividend yield and P/E ratio can affect the option price.

Dividend yield has been introduced by Merton as a parameter for option pricing model [4] and many researchers paid their attention on stochastic dividend yield for establishing more realistic option pricing model [9] or even with the reckoning of the stochastic market price of risk [38, 39]. Nevertheless, the P/E ratio is reasonably a significant parameter for option pricing models; as a result, we propose the extended model with consideration of the information of the P/E ratio.

In order to accomplish the new idea of taking the P/E ratio into account, we propose a new concept of stochastic process, namely the generalization of the Ornstein-Uhlenbeck process, and we name this process as GOU process. With the GOU process, we can consider the new parameter, the P/E ratio, in the option pricing model. Because of mathematical reason, we instead apply the reciprocal of P/E ratio, well known as earning yield. More realistic pricing model requires more realistic properties of the financial market. We follow Lioui's approach as the guiding path to construct the stochastic pricing model [38, 39]. Hence, the pricing model for the European call and put options taking into account stochastic dividend yield, stochastic market price of risk and stochastic earning yield are proposed in this thesis with the detailed proofs including the propositions for Greek's parameters. The proofs of these propositions in this thesis present the property of the strong stochastic behavior of the model which is very close to the real nature of the financial market.

To verify our extended option pricing model, we perform sensitivity analysis of the parameters in our model. It has been shown that not only the common parameters for pricing model, such as the volatility of the underlying stock (σ_S) and time to maturity (T), affect the model, but also the friction coefficient θ_δ and θ_ξ make an enormous effect to our option pricing model. In other words, the friction coefficients can be referred to the circumstance of the market at each point of time. The market price of risk parameters have the slightest impact on the option price; however, still can change the option prices. Subsequently, we observe the model characteristics by assuming the value of parameters. Noticeably, our option pricing model has the same basic trait and trend as the other well-known models but with the outstanding feature of stochastic property. The option prices of our model are extremely varied based on the parameters setting.

We examine our model by comparing the model results with the real world market data. Four stock indices, 1/100 Dow Jones Industrial Average (ˆDJX), S&P 500 (ˆGSPC), Russell 2000 (ˆRUT) and NASDAQ-100 (ˆNDX), have been chosen for the investigation because of their European option attribute. The simulation results show good agreement between our model results and the real-market data for both call and put options with the least sum of squared errors. In other words, our extended option pricing model provides fair option prices closest to the real market data in comparison with the other three existing models.

In summary, our mathematical option pricing model taking into account stochastic earning yield, stochastic market price of risk and stochastic earning yield is the seriously outstanding candidate for describing the real-world financial market.

For the future work, the parameter should be analyzed for some stock market indices. GOU processes may be proved for some theorems. Moreover, GOU process may be applied to the other research fields and tested for the possibility of other applications.

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